Scaling the Universe via a Transit of Venus

On 3 June 1769 Captain Cook observed the Transit of Venus in Tahiti. The intention was to use the observations to obtain an accurate estimate of the distance of the Earth from the Sun. This document outlines the calculations that might have been planned.

Knowns and Unknowns

Thanks to Kepler and others, there was, by Cook's time, a good understanding of the orbits of the planets. It was known that orbits were elliptical and the eccentricities were known too, as were the inclinations of the orbits to the plane of the ecliptic.

Given a specified time, the positions of the planets relative to the Sun and relative to one another could be predicted fairly precisely. The great unknown was the scaling factor. There was considerable uncertainty about any linear measurements across space.

A standard unit of distance used by astronomers is the semi-major axis of the Earth's orbit, the Astronomical Unit or AU. How could this be determined?

Triangulation

A naïve approach to measuring astronomical distances is to employ triangulation. The circle in Fig. 1 represents the Earth whose centre is at E. Two widely-separated observers at A and B are simultaneously looking at the centre of Venus at V.

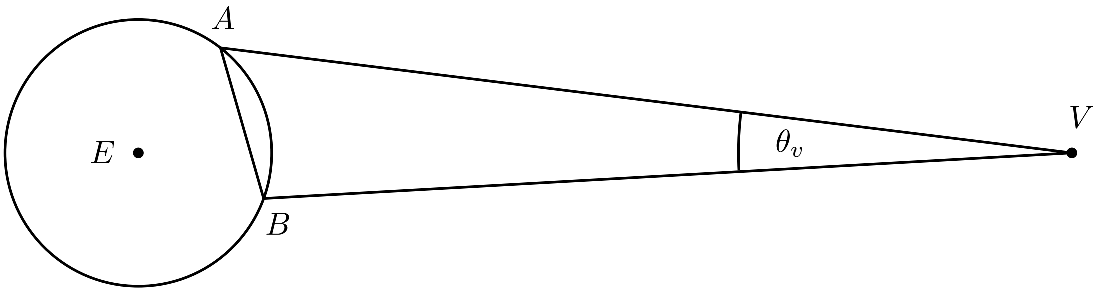

Fig. 1 — Using Triangulation

Suppose that E, A, B and V are in the same plane. If the geographical positions of A and B are known, the distance AB can be calculated and, with a little effort, the angles at A and B in the triangle ABV can be determined too. Given two angles and a side, the triangle can be solved and the distance from the Earth to Venus can therefore be established.

The difficulty is that Venus is so far away that the angle AVB, shown as θ_v, is close to zero. Typically it will be a fraction of an arc-minute. The lines AV and BV are almost parallel and the sum of the angles at A and B is close to $180°$.

It requires only the slightest errors in the measurements of the angles at A and B for the sum of these angles, as measured, to exceed $180°$.

This is therefore not a good method and any estimate of the distance of the Earth from Venus derived in this way will be unsafe.

Using a Transit of Venus

The geometry of Fig. 1 presents the problem and, in Captain Cook's day, it was thought that observing a Transit of Venus would provide a solution. During such a Transit, the observers at A and B would each see Venus crossing the Sun but the apparent paths taken would differ slightly and this difference could be used to determine θ_v.

Fig. 2 shows the general arrangement of the Earth (centre at E) and the Sun (centre at S) during a Transit of Venus (centre at V). The plane of the figure is that defined by the axis of the Earth (shown as an inclined broken line whose upper end is the north pole) and the centre of the Sun S.

In the course of the Transit, Venus moves from below the plane of the figure to above that plane in a direction which is not quite normal to the plane. At the instant represented, V is in the plane.

Points A_e and B_e (on the Earth) are as points A and B in Fig. 1 and, to make the calculations easier, these two points are on a common line of longitude where local sun time is noon in the figure. Accordingly, A_e and B_e are also in the plane of the figure.

The two lines A_eV and B_eV are produced and intersect the surface of the Sun at A_s and B_s respectively. At the instant represented, an observer at A_e sees Venus apparently crossing the Sun at A_s and an observer at B_e sees Venus apparently crossing the Sun at B_s.

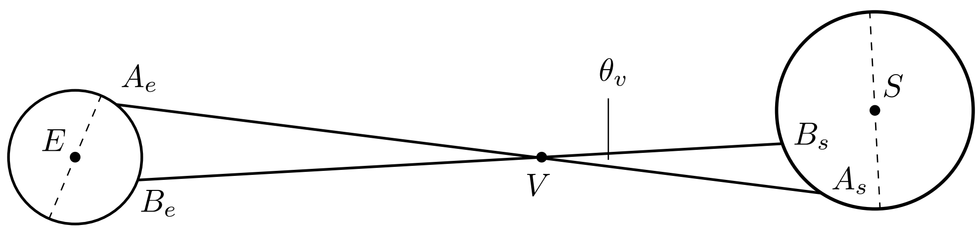

Fig. 2 — A Transit of Venus

The observers have no strong sense of the Sun being approximately spherical. Each will perceive the Sun as a circle, being the cross-section of the Sun normal to the observer's line of sight. This circle is the solar disc and the broken line through S is the approximate position of the diameter of this disc that lies in the plane of the figure. Strictly, the observers see two different solar discs but their angular separation is negligible.

As seen by the observer at A_e, the path apparently traced by Venus across the Sun is further from the centre of the solar disc than the path seen by the observer at B_e. Accordingly, the path seen from A_e is shorter and takes less time.

Timing the Transit is crucial to the method as a whole. By noting the difference in transit times recorded by the two observers, one could (it was hoped) compute a good estimate of the separation of the two paths.

An added complication is the rotation of the Earth. A Transit of Venus lasts about 6 hours during which the Earth rotates about 90°. In consequence, A_e and B_e are above the plane of the figure for the first three hours and below the plane for the last three hours. This rotation will be ignored for the moment.

The calculations make use of three distances:

$$r_e = ES \quad \text{the Centre of the Earth to the Centre of the Sun}$$
$$r_v = VS \quad \text{the Centre of the Venus to the Centre of the Sun}$$
$$r_d = EV \quad \text{the Centre of the Earth to the Centre of Venus}$$

These three values are not only unknowns but they are also variables. In particular, neither the Earth nor Venus is a constant distance from the Sun. The ratio r_v/r_e though can (and could) be predicted fairly precisely for any specified moment. Also, during a Transit, r_d is (almost exactly) the difference $r_e - r_v$.

None of the three values features directly in Fig. 2 and it is helpful to derive from that figure the geometrical construction which is shown in Fig. 3. In this, the distance r_e is taken as 1 unit and the values r_v and r_d are derived from the ratio and the difference.

An arc is drawn centre V and radius r_d and this passes through the centre of the Earth E. A second arc is drawn with the same centre V but radius r_v and this passes through the centre of the Sun S.

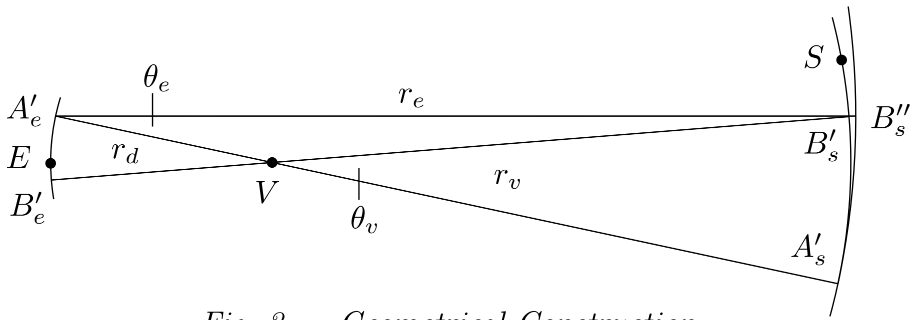

Fig. 3 — Geometrical Construction

The two lines $A'_eVA'_s$ and $B'_eVB'_s$ correspond to the lines A_eVA_s and B_eVB_s in Fig. 2 but have been extended to meet the two arcs. The angle of intersection between the two lines is again θ_v.

The third arc in Fig. 3 has centre A'_e and radius r_e. Since $A'_eV + VA'_s = r_d + r_v = r_e$ this arc passes through A'_s but it does not, quite, pass through B'_s. Instead it passes through B''_s which is the point where the construction line $A'_eB'_s$ produced meets the arc. Note that $A'_eB''_s = A'_eA'_s = r_e$.

An important angle is $A'_sA'_eB''_s$ which is shown as θ_e. It is related to θ_v:

$$\text{arc } A'_sB'_s \approx \text{arc } A'_sB''_s \qquad \text{so} \qquad r_e\theta_e \approx r_v\theta_v \tag{1}$$

Also:

$$\text{arc } A'_eB'_e = r_d\theta_v \approx (r_e - r_v)\theta_v \qquad \text{so} \qquad \theta_v \approx \frac{\text{arc } A'_eB'_e}{(r_e - r_v)} \tag{2}$$

Taking (1) and (2) together and treating approximations as equalities:

$$r_e\theta_e = r_v.\frac{\text{arc } A'_eB'_e}{(r_e - r_v)} = \frac{r_v}{r_e}.\frac{\text{arc } A'_eB'_e}{(1 - \frac{r_v}{r_e})} \qquad \text{so} \qquad r_e = \frac{r_v}{r_e}.\frac{\text{arc } A'_eB'_e}{(1 - \frac{r_v}{r_e})\,\theta_e} \tag{3}$$

The Distance of the Earth from the Sun

The right-hand side of (3) is an expression for the distance of the Earth from the Sun. Consider the various terms in that expression:

$$r_e = \frac{r_v}{r_e} \cdot \frac{\text{arc } A'_e B'_e}{(1 - \frac{r_v}{r_e})\, \theta_e}$$

The ratio r_v/r_e

As already noted, the ratio r_v/r_e (which appears twice in the expression) can and could be determined fairly precisely. Its value is about 0.72.

The arc $A'_e B'_e$

The principal difficulty when preparing diagrams of the Sun and planets is that, in most cases, scale drawings are impracticable. For example, the angle θ_v is about one arc-minute which means that the lines $A_e V$ and $B_e V$ are almost parallel.

Unlike the previous figures, Fig. 4 is close to being a proper scale drawing. This figure shows the Earth end of the lines from A_e and B_e to Venus. On this scale, Venus is over 100m off to the right.

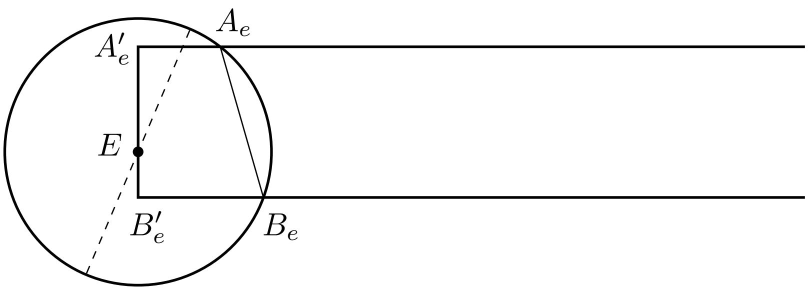

Fig. 4 — The Arc $A'_e B'_e$

The arc $A'_e B'_e$ which passes through the centre of the Earth is a very close approximation to a straight line. It is drawn vertically in Fig. 4, perpendicular to the parallel lines which are drawn horizontally.

The Earth's axis, the broken line through E, is inclined at just under 23° to the vertical which corresponds to the declination of the Sun at the time of the Transit on 8 June 2004.

Knowing the declination of the Sun and the latitude of A_e and B_e, it is not hard to determine $A'_e B'_e$, the arc $A'_e B'_e$ in the expression.

The angle θ_e

The angle θ_e is the outstanding term in the expression. Determining its value requires the observations provided by the observers at A_e and B_e to be analysed together.

In Fig. 2 the observer at A_e sees the Sun quite low in the sky to the south and the observer at B_e sees the Sun quite high in the sky to the north. The observers will have different perceptions about which point on the solar disc is uppermost. There is a need to agree a common orientation.

A standard approach is to regard the solar disc as a small circle marked out on the celestial sphere. Fig. 5 shows the solar disc as it might be drawn on a chart, with the north celestial pole in the direction indicated by N.

The broken line which forms the vertical diameter corresponds to the broken line through S in Fig. 2. This is in the same plane as (but not parallel to) the Earth's axis and is part of a great circle which extends from the north celestial pole to the south celestial pole.

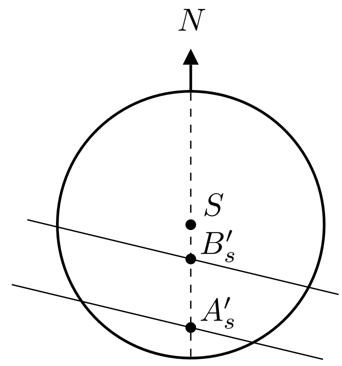

Fig. 5 — The Solar Disc

Suppose that the two Earth-bound observers at A_e and B_e are each asked to sketch the apparent passage of Venus across the solar disc. With a considerable amount of wishful thinking, one may imagine that when the two sketches are superimposed they appear as the inclined lines in Fig. 5.

In each case, the passage of Venus is from left to right along a path which slopes slightly downwards. In the Transit of 8 June 2004, the slope was about $13\frac{1}{2}^\circ$.

At the instant Venus crosses the plane defined by the Earth's axis and the centre of the Sun (the plane of Figs 2 and 3), the observer at A_e records Venus at A'_s and the observer at B_e records Venus at B'_s. These points on the solar disc are where the lines A_eV produced and B_eV produced meet the broken line through S in Fig. 2. In practice, they almost coincide with the points A'_s and B'_s in Fig. 3.

In the contrived circumstances of Fig. 5, the separation of the inclined lines is equivalent to a quarter of a solar diameter. On the celestial sphere, all distances are given as angles and, taking the diameter of the solar disc as $32'$, the observers conclude that their two paths are $8'$ apart.

The angular separation of A'_s and B'_s is the angle θ_e, the outstanding unknown in (3). Taking the separation of the lines as $8'$ and the common slope as $13\frac{1}{2}^\circ$, the result is:

$$\theta_e = \frac{8}{\cos(13.5)} \text{ arc-minutes } \approx \text{ 8.2 arc-minutes} \tag{4}$$

The true angular separation of A'_s and B'_s is much less than one arc-minute and the two inclined lines would be less than 1mm apart when drawn to scale in Fig. 5. Any attempt based on sketching apparent paths across the solar disc is destined to fail. There is a more precise way of estimating the separation of the paths...

Determining θ_e— a more refined approach

Many details about the 1769 Transit would have been known before Cook set sail. In particular, the time taken for Venus to cross the face of the Sun when viewed by an observer hypothetically placed *at the centre* of the Earth could have been predicted fairly precisely. For an outline of the procedure, see the Appendix at the end of this document.

The principal unknown (in some sense the object of the expedition) was the size of the Earth as a fraction of an Astronomical Unit. It would not therefore have been possible to predict how much effect the rotation of the Earth would have on the time of Transit recorded by an observer *on the surface* of the Earth.

Had Cook been observing the 2004 Transit, he would have carried with him the information which is summarised in Fig. 6. This shows, almost to scale, the apparent path of Venus across the Sun as viewed from the centre of the Earth. The north celestial pole is again off the top of the figure in the direction indicated by N.

The start of the path, marked as V_{12}, is shown as having a position angle of 119°. This angle is NSV_{12}, the angle of the line SV_{12} measured anti-clockwise round from SN. The end of the path, marked as V_{34}, has a position angle of 214°. This is the angle of the line SV_{34} again measured anti-clockwise round from SN.

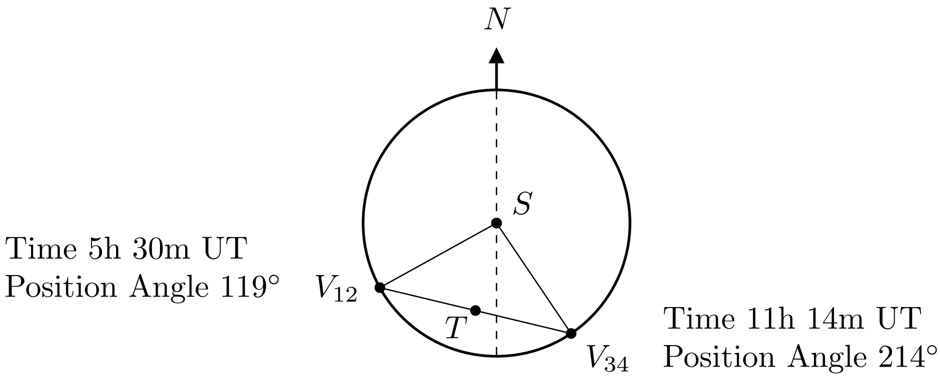

Fig. 6 — The Solar Disc on 8 June 2004

During a Transit there are four so-called contacts. First contact is when Venus first touches the rim of the solar disc and second contact is when Venus is just wholly inside the rim. Third and fourth contacts are the corresponding instants at egress. In the figure, V_{12} is the point on the rim of the solar disc marking the centre of Venus at a time halfway between first and second contact. V_{34} is the point halfway between third and fourth contacts.

From the two position angles it is straightforward to calculate the downward slope of the line $V_{12}V_{34}$ as $13\frac{1}{2}^{\circ}$.

Cook would also have known, fairly precisely, the local times (in Tahiti) that Venus would be expected at V_{12} and V_{34}. On 8 June 2004, these times in the U.K. were approximately 5h 30m UT and 11h 14m UT.

Point T on the transit path indicates the position of Venus in mid-Transit. Given that the path slopes downwards, this point is reached some time before the position angle of Venus is 180° (at the point of intersection of the transit path and the broken line).

Taking the (angular) radius of the Sun as 16 arc-minutes, and using the information just given, one can readily calculate the following:

Difference in Position Angle: $P_{34} - P_{12} = 214^\circ - 119^\circ$	95°
Angular distance $V_{12}V_{34} = 2 \times 16\ \sin(95/2)$	$23'\ 36''$
Angular distance $ST = 16\ \cos(95/2)$	$10'\ 49''$
Time of Transit: 11h 14m − 5h 30m	344 minutes

The difference in position angle $P_{34} - P_{12}$ of 95° can be re-expressed:

$$P_{34} - P_{12} = 2 \times \cos^{-1}\left(\frac{ST}{SV_{12}}\right) = 2 \times \cos^{-1}\left(\frac{10.81}{16}\right)$$

In this, $10.81'$ corresponds to $10'\ 49''$ and it is instructive to see the effect of reducing this value half an arc-minute to $10.31'$ and increasing it half an arc-minute to $11.31'$:

$P_{34} - P_{12}$	ST	$V_{12}V_{34}$	*Time*
$2 \times \cos^{-1}(10.31/16)$	$10'\ 19''$	$24'\ 28''$	356m 49s
$2 \times \cos^{-1}(10.81/16)$	$10'\ 49''$	$23'\ 36''$	344m 00s
$2 \times \cos^{-1}(11.31/16)$	$11'\ 19''$	$22'\ 38''$	330m 03s

The three entries in this table correspond to three views of the passage of Venus across the solar disc on 8 June 2004. The middle entry corresponds to the path shown in Fig. 6 and refers to the view from the centre of the Earth. Deem this to be the base case.

The first entry corresponds to a more southerly viewpoint where the path is displaced $30''$ towards the centre of the solar disc. The third entry corresponds to a more northerly viewpoint where the path is displaced $30''$ further from the centre.

Displacing the path $30''$ towards the centre extends $V_{12}V_{34}$ by $52''$ whereas displacing the path $30''$ away from the centre reduces $V_{12}V_{34}$ by $58''$. The difference is accounted for by the assumption that the solar disc is circular.

It makes little sense for an observer to attempt to measure the angular displacement $V_{12}V_{34}$ directly. The difference from the base case is too small. Measuring the time of Transit is altogether more profitable. Shifting one's viewpoint sufficiently to displace the transit path just half an arc-minute changes the time of Transit by nearly a quarter of an hour.

By extending the table of ST versus *Time* an observer would be able to estimate ST to high precision. An error in the transit time of one minute would lead to an error in the estimate of ST of about two arc-seconds. The organisers of Cook's expedition hoped to be able to estimate the time of Transit to a couple of seconds (of time!).

Using the time of Transit provides a good value of ST. The hypothetical observers can now report their separate estimates of ST. The difference between the two values can be used as a substitute for the top line in (4) to give what is now a good estimate for θ_e:

$$\theta_e = \frac{ST_B - ST_A}{\cos(13.5)} \qquad (5)$$

Tail-piece

In the interests of keeping the foregoing discussion within reasonable bounds, a few matters have been either over-simplified or overlooked.

For example, angles have generally been quoted in degrees whereas many of the expressions require the use of radians. Most notably, θ_e in (3) should be in radians.

The information about the 2004 Transit, presented in Fig. 6, is taken from the 2004 edition of *Whitaker's Almanack*. Almost certainly the position angles and times are appropriate for an observer in the U.K. and not for a hypothetical observer at the centre of the Earth. The procedure is sound even if the data are not quite appropriate.

A minor matter is that θ_e as originally defined and as used in (3) is the angle subtended at point A'_e in Fig. 3 which is not quite the position of the observer who is at point A_e in Fig. 2. Compared with the distance of the Earth to the Sun, the difference in position of A_e and A'_e is negligible so this issue may safely be ignored.

A much more significant shortcoming is a consequence of the Earth rotating about a quarter of a revolution during the course of a Transit of Venus. As the Earth travels along its orbit, an observer on the equator at noon sustains a retrograde motion which subtracts a small amount from the orbital speed. This has the effect of reducing the apparent Transit time by the order of 5 to 10 minutes.

It would have been difficult to take this into account in Cook's day. One approach would have been to have two observers on the same longitude but at equal and opposite latitudes. The effect would then have been the same for both. It is, of course, essential to choose locations where the entire transit can be observed otherwise it cannot be timed.

Alternatively, the effect could be ignored initially and then, after a provisional estimate of the Astronomical Unit had been determined, the effect could be taken into account and the estimate revised.

Cook himself encountered another difficulty. When observing the beginning and end of the Transit he and his colleagues were unable to agree on the precise times of the contacts. The problem was what is sometimes called the black-drop effect. The instants, particularly of the second and third contacts, are not very clear.

Following the 1769 Transit, around 600 papers on the subject were presented to the Royal Society who had sponsored Cook's expedition. A particularly impressive analysis of the observation data from five different locations (including Cook's in Tahiti) was made by Thomas Hornsby of Oxford University. This gave a value for the Earth–Sun distance which was within one percent of today's accepted value and a definite improvement on previous estimates.

Further historical details of Cook's voyage can be found via:

`http://transitofvenus.auckland.ac.nz/explorations`

F.H. King
21 August 2004

Appendix — Forecasting the Transit Time

This appendix provides further understanding of the orbits of the Earth and Venus and outlines a simple approach to estimating the transit time. There is no attempt to provide a full analysis.

Coplaner Orbits

Fig. 7 shows a hypothetical case where, at mid-Transit, the centres of the Earth, Venus and the Sun (E_0, V_0 and S respectively) exactly align. An observer at the centre of the Earth would see Venus in the centre of the solar disc.

In a grossly exaggerated way, points E_1 and V_1 mark the positions of the Earth and Venus one hour later. Both bodies have moved along their orbits but Venus has moved rather more than the Earth.

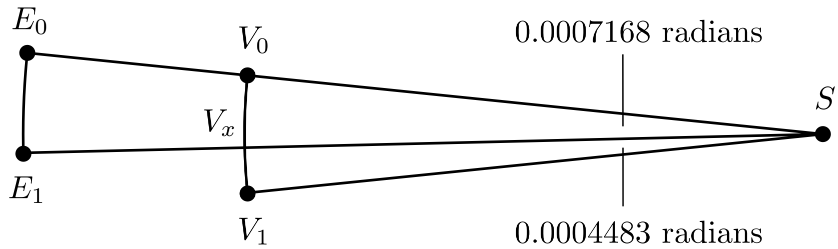

Fig. 7 — One Hour Segments of Two Coplaner Orbits

As a first step, suppose that the orbits of the two planets are circular and coplanar and use the following data:

	Radius of Orbit	*Orbital Period*
Earth	1.0000 AU	365.25 days
Venus	0.7232 AU	224.7 days

The radius of the Earth's orbit, E_0S, is given as 1 Astronomical Unit, 1 AU, and that of Venus, V_0S, is given as 0.7232 AU. This fraction would have been known to a fair approximation in Cook's day as would the orbital periods of the two bodies (measured in Earth days). The great unknown was how many miles there were in 1 AU.

It is easy to determine the angular displacement of each body around its orbit in one hour:

$$\text{Earth} = \frac{2\pi}{24 \times 365.25} = 0.0007168 \text{ radians} \tag{6}$$

$$\text{Venus} = \frac{2\pi}{24 \times 224.7} = 0.0011651 \text{ radians} \tag{7}$$

The angular displacement of the Earth, 0.0007168 radians, is shown in the figure as is the difference between the two displacements, 0.0004483 radians.

An Earth-bound observer would not readily appreciate that in one hour Venus moves from V_0 to V_1. Given the movement of the Earth, the appearance would be of movement from the point shown as V_x in the figure to V_1. To determine this as an apparent angular movement across the solar disc, first compute:

$$V_x V_1 = 0.7232 \times 0.0004483 = 0.0003243 \text{ AU}$$

$$E_1 V_x = 1.0000 - 0.7232 = 0.2768 \text{ AU}$$

The apparent angular displacement in one hour is therefore:

$$\tan^{-1}\left(\frac{0.0003243}{0.2768}\right) = 0.001172 \text{ radians} \tag{8}$$

Expressed in arc-minutes:

$$\text{Angular movement } = 60 \times \frac{180}{\pi} \times 0.001172 = 4.03 \text{ arc-minutes per hour} \tag{9}$$

Using the data given previously for the 2004 Transit, the transit path was $23.59'$ and this took 344 minutes which translates into 4.12 arc-minutes per hour. The value given in (9) is too low.

Different orbital planes

The orbits of the Earth and Venus are not coplanar. The two orbital planes are inclined at an angle of about 3.4° one relative to the other.

The line of intersection of the orbital planes of two planets is known as the line of nodes. Each planet crosses this line twice during an orbit. A Transit occurs when the two planets cross the line of nodes almost simultaneously and on the same side of the Sun.

In the hypothetical circumstances of Fig. 7, the line of nodes is E_0V_0S. The plane defined by the points E_0, E_1 and S is the plane of the Earth's orbit and point V_1 is not in this plane. The view from E_0 looking towards the Sun is approximately as illustrated in Fig. 8.

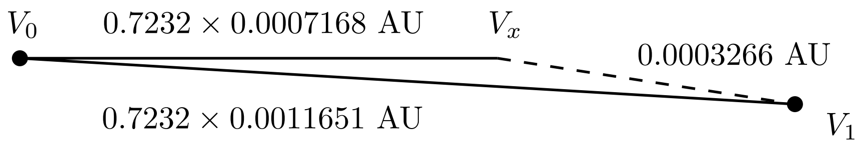

Fig. 8 — One Hour Segments of Two Non-Coplaner Orbits

In the figure, line V_0V_x is in the plane of the Earth's orbit and corresponds to the angular displacement of the Earth in one hour, 0.0007168 radians [from (6)]. The line V_0V_1 is inclined at 3.4° to V_0V_x and corresponds to the angular displacement of Venus in one hour, 0.0011651 radians [from (7)].

To derive the absolute distances V_0V_x and V_0V_1, the angles have to be multiplied by the distance of Venus from the Sun, 0.7232 AU. Viewed from the Earth, Venus appears to move from V_x to V_1. Using the Cosine Rule, this displacement is about 0.0003266 AU, as shown in the figure.

The apparent angular displacement in one hour is therefore:

$$\tan^{-1}\left(\frac{0.0003266}{0.2768}\right) = 0.001180 \text{ radians}$$

Expressed in arc-minutes:

$$\text{Angular movement } = 60 \times \frac{180}{\pi} \times 0.001180 = 4.06 \text{ arc-minutes per hour} \qquad (10)$$

Fig. 8 has been drawn using ecliptic coordinates and the line V_0V_x which has been drawn horizontally is in the plane of the Earth's orbit, the Ecliptic plane. The apparent angular displacement of $4.06'$ can be resolved horizontally and vertically:

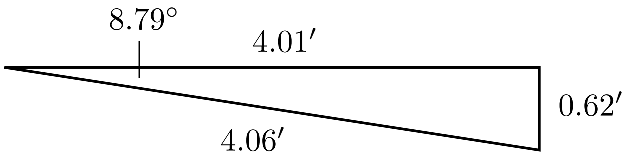

Fig. 9 — Ecliptic Coordinates

The horizontal component, $4.01'$, is the relative change in celestial longitude in one hour. The vertical component, $0.62'$, is the relative change in celestial latitude in one hour.

At alignment, when viewed from the centre of the Earth, the centres of the Sun and Venus coincide on the celestial sphere. The Sun and Venus have the same latitude and longitude.

Assuming a plane circular orbit, the Sun's latitude is always zero and its longitude increases at a constant rate. The motion of Venus is much less regular. For the first hour after alignment, the longitude and latitude both decrease. The net change in longitude of $4.01'$ is the combination of the increase of the Sun's longitude plus the decrease in Venus's longitude. The net change in latitude of $0.62'$ is due to Venus alone.

The downward angle of 8.79° shown in Fig. 9 is less than the downward angle of $13\frac{1}{2}^\circ$ in Fig. 6. The difference is largely accounted for by Fig. 6 being drawn using equatorial coordinates. Any line drawn horizontally in Fig. 6 is parallel to the celestial equator, not parallel to the ecliptic.

The 2004 Transit was about two weeks before the summer solstice. Using equatorial coordinates, the Sun was still heading northwards. Viewed from the Sun, the Earth was heading southwards with a downward slope of about 5.70°. Adding this value to 8.79° gives a combined result which exceeds the angle of $13\frac{1}{2}^\circ$ in Fig. 6.

The net angle of slope is a little too large and the value given in (10) is a little too low. The model needs further enhancement. The most important omission is that no account is taken of the Earth's rotation.

The Effect of the Earth's Rotation

Imagine an observer on the surface of the Earth on the line E_0S in Fig. 7. The observer is displaced from E_0 towards S by the radius of the Earth. One hour later the centre of the Earth has moved to E_1 but the observer is not quite on the line E_1S. The Earth rotates 15° anti-clockwise so the observer will not have travelled quite so far as the centre of the Earth.

Accordingly, V_0V_x will be slightly reduced and V_xV_1 (Fig. 8) will be slightly increased. This will increase the value given by (10) but by an amount which depends on the Observer's latitude.

Further Refinements

Further refinements would include taking into account that the orbits are not circular and that centres of the Earth, Venus and the Sun do not ever exactly align. Real transits take place close to, but not actually on, the line of nodes.

The additional calculation required to refine the value given in (10) and to determine the position angles in Fig. 6 will not be considered here.