The Angles of the Time Lines on a Sundial

Telling the time by a sundial is very much like telling the time by an analogue clock. A typical such clock has flat circular face or *dial* on which there are various markings, some of which may be labelled with times. An hour hand and a minute hand rotate about the centre of the dial and their positions relative to the markings indicate the time.

The markings, or *time lines*, radiate from the centre of the dial but it would be unusual for any of these lines to be drawn all the way from the centre to the edge. On most clock face designs the time lines are marked as short lengths of line near the rim of the dial.

As with clocks, sundials commonly have flat dials marked with labelled time lines but, instead of looking at the positions of clock hands, the user of a sundial looks at the position of a single straight-line shadow and notes where this falls in the dial markings.

A conventional sundial is equipped with a *gnomon* which takes the form of a straight-edge or straight rod which is not in the plane of the dial. Since the gnomon is straight, its shadow will fall as a straight line and it is this shadow which indicates the time.

It is essential for the gnomon to be parallel to the axis of the Earth because, given such an arrangement, the direction of the shadow of the gnomon at a given *hour-angle* of the sun is independent of the *declination* of the sun.

Provided the gnomon and the dial are not parallel to one another they will intersect at some point, *the root of the gnomon*, which is the obvious point to use as the origin of any coordinate system set up to analyse a sundial. Clearly the shadow will radiate from the root of the gnomon and the dial markings radiate from this point too.

If the gnomon is parallel to the dial (which happens if the plane of the dial is parallel to the axis of the Earth), the gnomon has to be supported so that it is displaced from the dial. The time lines are drawn as a set of lines parallel to one another and parallel to the gnomon.

Design Considerations — Scope of this Document

There are many, many different kinds of sundial and this document is intended to serve as a design aid for one particular kind only. Three assumptions are made:

1. The sundial is intended for terrestrial use.
2. The gnomon is a straight line parallel to the axis of the Earth.
3. The dial is to be marked on a plane surface which has arbitrary orientation.

A designer will need to establish the latitude (ϕ) of the sundial and the orientation of the plane of the dial. The orientation will be expressed by two parameters, the azimuth (A) and lean (ℓ), where the azimuth is the direction faced by the plane of the dial relative to true North and the lean is the inclination of the plane to the vertical.

In the sections that follow, a suitable coordinate system will be described and formulae will be derived which will enable a designer to set up the gnomon and mark out the dial.

The Earth and the Sun

Any sundial is deemed to be attached to the Earth at some latitude and longitude. The Earth is assumed to rotate once every 24 hours relative to the sun. At any instant the line from the centre of the Earth to the sun intersects the Earth's surface at some point known as the *sub-solar point*. The latitude of the sub-solar point is the declination δ of the sun.

As the Earth rotates, the sub-solar point crosses every line of longitude. The instant the sub-solar point crosses the line of longitude shared by the sundial defines local noon at the sundial. The instant the sub-solar point crosses the line of longitude which is 180° displaced from the longitude of the sundial defines local midnight at the sundial.

At the sundial, the local hour-angle H of the sun is the number of degrees of longitude that the sub-solar point has crossed since local midnight. The hour-angle at midnight is 0° and the hour-angle at noon is 180°.

The line of longitude of the sundial together with the line of longitude 180° displaced define a great circle and the sun crosses the plane of this great circle twice a day, once at midnight and once at noon.

In sundial design it is usually convenient to think of the sun going round the Earth and to imagine the line from the centre of the Earth to the sun sweeping out a cone. The apex of the cone is at the centre of the Earth and the axis of the cone is coincident with the axis of the Earth.

At the Summer Solstice, when the declination is about $+23\frac{1}{2}^\circ$, this cone intersects the surface of the Earth along the Tropic of Cancer; the cone points towards the South Pole. At the Winter Solstice, when the declination is about $-23\frac{1}{2}^\circ$, this cone intersects the surface of the Earth along the Tropic of Capricorn; the cone points towards the North Pole. At the Equinoxes the declination is zero and the cone reduces to a disc which is in the plane of the equator.

The sun can be regarded as infinitely distant so all lines from any point on the Earth to the sun are deemed parallel. In consequence, a line drawn from an arbitrary point on a gnomon to the sun also sweeps out a cone during the course of a day and, since the gnomon is parallel to the axis of the earth, the axis of this cone is coincident with the line of the gnomon. The half-angle of the cone is the complement of the declination of the sun.

Mathematical Assumptions

Mathematically, any sundial of the kind being considered consists of three components:

1. A line. Mathematically, this has indefinite length. In practice, the gnomon will lie along a short stretch of it.
2. A plane. Mathematically, this has no thickness but indefinite extent. In practice, (unless the line and plane are parallel) dial markings will be drawn on this plane in the vicinity of the point where the line and plane intersect.
3. A cone whose apex is an arbitrary point on the gnomon and whose axis is the line of the gnomon. Mathematically, this is the figure generated as the line from the point on the gnomon to the sun sweeps round during the course of a day.

It will be assumed that the plane and the gnomon are opaque but that nothing else can interrupt sunlight reaching the sundial. Not only are there no clouds or local obstructions but the Earth itself is deemed to be transparent. The sun continues to shine on the sundial even when it is below the horizon.

Note that although a plane has no thickness it has two sides. Moreover the line of the gnomon continues through the plane so that it sticks out on both sides. One might imagine the finished sundial having a dark glass dial with the markings drawn in the thickness of the glass and a straight rod passing through the glass.

Any sundial is really two sundials in one. If the design requirement is for a North-facing sundial then the North side will be regarded as the *obverse* side but mathematical analysis will simultaneously lead to a perfectly good South-facing sundial on the *reverse* side.

The serious reason for all these assumptions is to ensure that there is always a shadow to indicate the time. Given the absence of obstructions the sun will always shine on one side of the plane or the other and, given a push-through gnomon, there will always be a shadow.

In the imaginary sundial the shadow, like the markings, should be considered to be in the thickness (or thinness) of the dark glass. This glass should not be quite opaque so that the shadow can be seen from either side.

It is important not to overlook a special case: those instants when the sun moves from one side of the plane to the other and is momentarily in the plane of the dial. Since the real sun has a non-zero angular diameter it will theoretically shine on both sides of the dial at these instants so there will still be a shadow.

A System of Coordinates

When analysing any particular sundial, it is expedient to begin by setting up a right-handed (x, y, z) system of rectangular coordinates. In this document, whatever the orientation of the dial, the coordinates will be arranged thus:

1. The x-y plane will always be taken as the plane of the dial.
2. The x-axis will always be horizontal.
3. Unless the plane as a whole is horizontal, the y-axis will lie along the line of greatest slope and the positive y-axis should slope upwards. If the plane is horizontal the positive y-axis should be aligned either due North or due South. (It is usually more convenient to align the positive y-axis due South in the northern hemisphere.)
4. The z-axis is orthogonal to both the x- and y-axes and is therefore normal to the plane. Moreover, it is the *outward* normal of the *obverse* side of the dial.
5. The origin of the coordinate system should be chosen to coincide with the point where the line of the gnomon and the plane of the dial intersect.

Given the foregoing one can now consider a sequence of examples of graded complexity.

Example I — The Simplest Case

If a sundial is to be constructed on an arbitrary plane surface (such as a sloping roof) one of the most difficult tasks is accurately to determine the three principal parameters, latitude, azimuth and lean. It is usually straightforward to determine the latitude from a map but determining the orientation of the plane requires surveying skills which are not described in this document. In the discussions that follow, the parameters will simply be taken as given.

In the simplest possible case $\phi = A = \ell = 0°$.

Viewed practically, one can readily imagine a vertical wall lying in an East-West plane on the equator and being required to construct a sundial on the North side of this wall.

The designer needs to know how to orientate the gnomon and how to mark out the dial. . .

Example I — The Gnomon

Viewed mathematically, the specified plane is the equatorial plane and, since the axis of the Earth is normal to that plane, the gnomon will be normal to the plane of the dial. The designer has a particularly easy task in this case but it is worth taking some care over the analysis in readiness for more ambitious examples.

First set up the coordinates. The x-y plane is the plane of the dial and, viewed from the North, the x-axis runs horizontally from left to right (East to West), the y-axis runs vertically upwards, and the z-axis is horizontal and points due North. In this special case the line of the gnomon coincides with the z-axis.

Also in this special case, the y-z plane coincides with the plane defined by the local line of longitude and the line of longitude 180° displaced. The sun crosses this plane at midnight (when its y coordinate is negative) and again at noon (when its y coordinate is positive).

In the general case, it will be useful to the designer to be told the orientation of the gnomon relative to the plane of the dial. It is usual to specify what angle the *projection* of the gnomon onto the dial makes with the y-axis. This projection is the appearance of the gnomon when the wall is viewed in elevation. In the present case the projection of the gnomon onto the dial is simply a point at the origin.

It is also useful to the designer to know the angle the gnomon makes to the normal to the wall or the complement of this angle which is, in general, the angle the gnomon makes with its projection. In the present case the gnomon is normal to the wall so the angle to the normal is 0° and the complement of this angle is 90°.

One way to determine the direction of the shadow of the gnomon on the dial is to consider the position of the shadow of an arbitrary reference point on the line of the gnomon. In a real sundial the gnomon will have finite length and it is convenient to use the tip of the gnomon as the reference point. Suppose the length of the gnomon is g (where $g > 0$). The coordinates of the tip of the gnomon are:

$$\text{Tip} = \begin{pmatrix} x_T \\ y_T \\ z_T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \tag{1}$$

Remember, mathematically, that the line of the gnomon runs through the dial and out the other side. It is convenient to imagine the gnomon having two ends; the North end runs from the origin to the North tip and the South end runs from the origin to the South tip. In general, the coordinates of the South tip will be the same as those of the North tip but each reversed in sign. In the present case the coordinates are:

$$\text{Tip} = \begin{pmatrix} x_T \\ y_T \\ z_T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

Example I — The Frame

In sundial design, it is angles which are important rather than lengths. The frame or outline of the dial can be any shape the designer wishes and the root of the gnomon can be anywhere within the frame and, in some designs, is actually outside the frame.

The value of g is not critical but it is usual to make a gnomon long enough for its shadow at least to reach the edge of the dial most of the time that there is a shadow. Very commonly, the length of the gnomon is roughly half the maximum dimension of the frame.

Example I — The Dial Markings

It has been noted that the direction of the shadow of the gnomon on the dial can be determined by considering the position of the shadow of the tip. More strictly, one should consider the position of the shadow of the *appropriate* tip. It is important to choose the tip which, at a given time, is on the same side as the sun.

The *position* of the shadow of the (appropriate) tip depends on four things:

1. The hour-angle of the sun.
2. The declination of the sun.
3. The coordinates of the tip.
4. The orientation of the plane.

Happily the *direction* of the shadow on a particular dial depends only on the hour-angle and so can be used to determine the time.

In the particularly simple case under consideration, the sun is on the North side of the plane when the declination is positive (from roughly 21 March to 23 September) and on the South side when the declination is negative (the rest of the year). It seems sensible to analyse the two cases, $\delta > 0$ and $\delta < 0$, separately.

Start with the case $\delta > 0$ when the sun is on the North side of the plane and the appropriate tip has coordinates $(0, 0, g)$. At midnight, when $H = 0$, the sun is in the y-z plane. As previously noted the sun has a negative y coordinate at midnight but its z-coordinate is positive given that the declination is positive.

Accordingly, at midnight, the extension of the line from the sun to the tip at $(0, 0, g)$ intersects the x-y plane on the positive y-axis at $(0, g/\tan\delta, 0)$. The y-axis is in the equatorial plane and the line from the sun makes an angle δ to this axis.

Given that $g > 0$ and $\delta > 0$ (in this case) the y coordinate, $g/\tan\delta$, of the shadow of the tip is positive. Note that $\delta < 0$ would lead to a silly result. The y coordinate would be negative. The sun would be on the wrong side of the plane and instead of the line from the sun to the tip continuing onto an intersection point on the plane, the intersection point would be between the sun and the tip.

In the case $\delta < 0$ the sun is on the South side of the plane and the appropriate tip has coordinates $(0, 0, -g)$. At midnight the z coordinate of the sun is negative and the line from the sun to the tip now intersects the y-axis at $(0, -g/\tan\delta, 0)$. The y coordinate is once again positive as required at midnight.

Now consider the position of the shadow of the (appropriate) tip at general hour-angle H. Start again with the case $\delta > 0$ with the sun on the North side. Consider the line from the sun to the North-side tip over the course of a day.

The extension from this tip to the plane of the dial generates another cone (one that points North) and since the plane is normal to the gnomon the intersection of this cone and the plane will be a circle whose radius is $g/\tan\delta$. Given that $g > 0$ and $\delta > 0$ the radius is positive.

This circle is the path traced by the shadow of the tip of the gnomon during the course of a day. This circle turns out to be exceedingly useful, not only in the present case but also in the analysis of the general case when the plane of the dial has arbitrary orientation.

The position on the circle of the shadow of the tip depends on the hour-angle of the sun. Given the symmetry of the case under consideration, the shadow moves at a uniform rate round the circle. This circle can be expressed mathematically using the hour-angle H as a parameter:

$$\text{Circle} = \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} = \begin{pmatrix} \sin H \\ \cos H \\ 0 \end{pmatrix} \frac{g}{\tan\delta} \tag{2}$$

These are the coordinates of the shadow of the North tip of the gnomon at hour-angle H. Since the circle lies in the x-y plane the z coordinate is zero. As required, the shadow lies along the positive y-axis at midnight and this will be taken as the reference direction.

Any time line is fully specified by the direction of the line from the root of the gnomon to the shadow of the tip at the time in question. The direction is conveniently given as the angle made to the reference direction. Calling this angle θ, its value at hour-angle H is:

$$\tan\theta = \frac{x_C}{y_C} = \frac{\sin H \times g/\tan\delta}{\cos H \times g/\tan\delta} = \frac{\sin H}{\cos H} \tag{3}$$

The analysis has assumed that $g > 0$ and $\delta > 0$ so it is legitimate to cancel $g/\tan\delta$ from the top and bottom lines and so derive the result which shows that θ is independent of g and δ. Accordingly, the same set of time lines will be appropriate for any positive length of gnomon and any positive declination of the sun.

The analysis of the case $\delta < 0$ follows the same lines. The sun is on the South side of the plane so use is made of the tip on the South side. The extension of the line from the sun to this tip generates a (South-pointing) cone and the intersection of this cone and the plane is again a circle of whose radius is $-g/\tan\delta$. Given that $-g < 0$ and $\delta < 0$ the radius again is positive.

The only revision to (2) reflects the change in the z coordinate of the tip from g to $-g$:

$$\text{Circle} = \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} = \begin{pmatrix} \sin H \\ \cos H \\ 0 \end{pmatrix} \frac{-g}{\tan\delta}$$

A reworking of (3) leads to the same result as before:

$$\tan\theta = \frac{x_C}{y_C} = \frac{\sin H \times (-g/\tan\delta)}{\cos H \times (-g/\tan\delta)} = \frac{\sin H}{\cos H}$$

Notice that the angle of a time line depends only on the hour-angle H and is independent of either g or δ.

Having determined the position of the shadow of the appropriate tip, the straight line shadow which indicates the time is simply the line from the root of the gnomon to the shadow of the tip. During the course of the day the shadow will sweep round the dial rather as a moving clock hand.

Every hour, on the hour, one could draw a line from the root of the gnomon along the line of the shadow and label the line with the time of day. Still ignoring the fact that the sun is below the horizon half the time, the 24 radiating lines would form the dial.

Example I — Two-Argument Inverse-Tangent Functions

Clearly, in the present case, $\tan\theta = \tan H$ so $\theta = H$ indicating that the shadow sweeps round the dial at a uniform rate (15° per hour). Hour lines could be marked on the dial at 15° intervals.

Determining an angle from its tangent requires the resolution of an ambiguity. For example, given that $\tan\theta = 1$ it is impossible to tell whether $\theta = 45°$ or $\theta = 225°$.

The ambiguity can often be resolved by separate consideration of the numerator and the denominator of an expression for the tangent of an angle. Many computer systems supply a two-argument inverse-tangent function. For example, the Excel spreadsheet system incorporates the function ATAN2 and could readily determine the value of θ by adapting (3) to:

$$\theta = \mathsf{ATAN2}(\cos H, \sin H) \tag{4}$$

Notice that the denominator is the first argument and the numerator is the second. When $H = 225°$, $\cos H = \sin H = -1/\sqrt{2}$ and the expression $\mathsf{ATAN2}(-1/\sqrt{2}, -1/\sqrt{2})$ correctly yields 225° and not 45°. The ATAN2 function also conveniently deals with the case when the denominator is zero. At 6 a.m. $H = 90°$ and $\theta = \mathsf{ATAN2}(0, 1) = 90°$.

The desire to exploit two-argument inverse-tangent functions explains why the numerator and denominator in (3) are kept separate and the expression is not reduced to $\tan H$.

Example I — Negative Declination

An awkward difficulty when exploiting two-argument inverse-tangent functions is to ensure that the numerator and denominator are correctly signed in all circumstances.

Consider again the legitimacy of cancelling $g/\tan\delta$ from the top and bottom lines of (3). Depending on whether or not this common factor is cancelled, one obtains two expressions for θ:

$$\theta = \mathsf{ATAN2}(\cos H \times g/\tan\delta, \sin H \times g/\tan\delta) \qquad \text{and} \qquad \theta = \mathsf{ATAN2}(\cos H, \sin H)$$

Given the assumptions that $g > 0$ and $\delta > 0$, these expressions give the same result, but if $\delta < 0$ the signs of the two arguments are reversed and the results given for θ differ by 180°. Happily, this possibility cannot arise because when $\delta < 0$ the common factor to be cancelled is $-g/\tan\delta$ which is positive ensuring that cancelling does not change the signs of the numerator and denominator.

In more ambitious cases careful attention will have to be paid to such cancellation. In this simple case, the sign of the z coordinate of the tip of the gnomon is guarenteed to match that of $\tan\delta$ and all is well.

The only outstanding problem is cancelling when $\delta = 0$. Strictly, this is not permitted but, as explained earlier, when the declination is zero one can argue that the angular diameter of the sun is non-zero and assume that half the sun shines on one side of the plane with small positive declination and the other half shines on the other side with small negative declination. At a given hour-angle the results are the same and are as given by (3).

Example I — The View from the Reverse Side

In general, the mathematical analysis treats the plane of a dial as two-sided and regards the dial markings and the shadow to be in the thickness of the plane. The dial and shadow can be viewed from either side at all times.

For an arbitrarily oriented plane, the side of which the positive z-axis is an outward normal is definitively the obverse side. If this outward normal points North of the celestial equator, the shadow of a sundial constructed on the obverse side will rotate clockwise, otherwise it will rotate anti-clockwise. On the reverse side the sense of rotation is reversed.

In the specification of the simplest case, the plane of the dial coincides with the equatorial plane and the positive z-axis points North indicating that the North side of the plane is the obverse side. The shadow on the North side sweeps round clockwise and the shadow on the South side sweeps round anti-clockwise.

On the reverse side the x- and z-axes are reversed. Thus on the South side of the simple sundial the positive x-axis runs to the left and the positive z-axis runs into the dial rather than out of it. If in the original specification the South side of the plane had been deemed the obverse side, the coordinate system would have arranged for the positive x-axis to run to the right (East) and the z-axis to be an outward normal pointing South.

Subsequent analysis would have led to a modified expression for θ:

$$\theta = \mathsf{ATAN2}(\cos H, -\sin H) \tag{5}$$

This differs from (4) by reversing the sign of the numerator to reflect the reversal of the x-axis. Note that θ is still measured clockwise but the reversal of sign ensures that $H = -\theta$ so, when the South side is the obverse side, the shadow continues to be observed to sweep round anti-clockwise.

Example I — Guidelines for the Designer

Assuming that the sundial of Example I really is intended for a North-facing wall on the equator, the designer now has all the information needed to embark on construction. Here is a summary of the design stages which might now follow:

1. Choose a position on the wall for the root of the gnomon.
2. Use this point as an origin and draw x- and y-axes through the point. The positive y-axis should be vertical. This is the reference direction.
3. Mount the gnomon so that it is normal to the wall and its root is at the origin.
4. Decide on a frame or outline for the dial and for each required time line note the hour-angle H and use (3) to determine the value of θ. Mark the time line along the line from the root of the gnomon at an angle θ to the reference direction.

On a real sundial, time lines are typically marked at values of H corresponding to hours which fall in daylight, from 6 a.m. to 6 p.m. in the present case. It would rarely be sensible to draw all 24 hour lines but it is common to add markings to indicate selected half- and quarter-hours.

Example I — Translation

Suppose an English tourist takes a liking to the glass sundial, discovers that copies are available in a local souvenir shop and buys one. Will the sundial be of any use back in England?

In principle, a sundial which has been designed for a particular place can be set up to function anywhere else. All the new owner has to do is to be very careful about the orientation.

One can imagine transporting a sundial from one place to another in two stages:

1. Take the sundial to a place which has the same longitude as the original position but at the new latitude. Orientate the sundial so that the x-, y- and z-axes are each parallel to the way they were in the original location. Relative to the axes, the direction of a line to the sun at a given hour-angle and declination is exactly as it was and the sundial performs just as before.
2. Conceptually, detach the sundial from its mounting and allow the Earth to rotate underneath the sundial until the place with the desired longitude appears. Remount the sundial in its new position. While the Earth is rotating, keep the orientation of the sundial with respect to the sun fixed. This procedure is akin to stopping a clock for a while. After remounting, the sundial will rotate with the Earth again and the time indicated will be appropriate local sun time.

Suppose the glass sundial is to be set up at latitude 52° North. In the new location the plane of the dial has to be parallel to the equatorial plane and will therefore appear to lean to the South with an angle of 52°. The parameters for such a sundial would be $\phi = 52°$, $A = 0°$ and $\ell = -52°$ but this is exactly equivalent to the original sundial as far as orientation of the gnomon and placement of dial markings are concerned.

If moved to the North Pole, the dial would be horizontal so only the original North side would be useful. At the South Pole only the original South side would be useful.

By an extension of this reasoning, a sundial designed for an arbitrary plane surface can be set up to function at the equator. This is of note because the mathematical analysis for a dial with arbitrary orientation is simplified if the plane is considered to be at the equator.

Example II — Arbitrary Latitude

In general, a wall sundial will not be on the equator, the obverse side of the dial will not face due North and might not be vertical.

In the simplest case, $\phi = A = \ell = 0°$. The next simplest case is to let the latitude ϕ be arbitrary but keep $A = \ell = 0°$. In the context of a wall sundial, the wall is no longer on the equator but one side still faces due North and the wall is still vertical.

The required analysis is most conveniently undertaken if one imagines translating the plane of the dial to a plane parallel to it (and on the same line of longitude) at the equator. One can then consider the original simplest case and determine what happens if the plane of its dial is rotated so as to be parallel to the plane at latitude ϕ.

Example II — The Gnomon

Having returned to the equator, the next task is to set up a system of coordinates for the new dial. Call the coordinates (x', y', z') and arrange for the origin to be coincident with the origin of the dial in the simplest case.

The x'-y' plane is the plane of the dial and, viewed from the North, the x'-axis runs horizontally from left to right (East to West) and is coincident with the old x-axis. The y'-axis runs upwards but now leans to the North making an angle ϕ with the old y-axis. In like manner, the z'-axis points downwards at and angle ϕ relative to the old z-axis.

Any point in the new coordinate system (x', y', z') is related to a point (x, y, z) in the previous system by a rotation matrix as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Note that the gnomon of the new dial is the original gnomon which was deemed to have two tips, the North tip at $(0, 0, g)$ and the South tip at $(0, 0, -g)$. In the new coordinate system, the coordinates of the North tip are:

$$\text{Tip} = \begin{pmatrix} x'_T \\ y'_T \\ z'_T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\phi \\ \cos\phi \end{pmatrix} g$$

It has been noted that the designer generally needs to know the orientation of the gnomon relative to plane of the dial. It is usual to specify the angle the projection of the gnomon onto the dial makes with the positive y-axis and the angle the gnomon makes with this projection. Call these angles *proj* and *ang*. In general, for the North end of the gnomon, the values can be derived from:

$$\tan(proj) = \frac{x'_T}{y'_T} \qquad \text{and} \qquad \sin(ang) = \frac{z'_T}{g} \tag{6}$$

Using the coordinates just derived for the North tip:

$$\tan(proj) = \frac{0}{g\sin\phi} \qquad \text{so} \qquad proj = \begin{cases} 0^\circ, & \text{if } \phi \geqslant 0 \\ 180^\circ, & \text{otherwise} \end{cases} \tag{7}$$

The two results simply reflect the fact that, on the North side of the wall, the gnomon points upwards (so the projection is along the positive y-axis) if the wall is in the Northern hemisphere and points downwards if the wall is in the Southern hemisphere.

On the South side of the wall the sign of g is negative and the results reverse.

Again using the coordinates for the North tip:

$$\sin(ang) = \frac{g\cos\phi}{g} = \cos\phi \qquad \text{so} \qquad ang = 90^\circ - |\phi| \tag{8}$$

The inverse-sine function does not suffer from the same ambiguity problem that applies to the single argument inverse-tangent function. It is permissible to cancel the common factor g whatever its sign. Moreover, since $-90^\circ \leqslant \phi \leqslant 90^\circ$, $0 \leqslant \cos\phi \leqslant 1$ so $0 \leqslant \sin\phi \leqslant 1$ and $0^\circ \leqslant ang \leqslant 90^\circ$ which justifies the use of the absolute value of ϕ. The result applies on both the North side and the South side and in both hemispheres.

Example II — The Dial Markings

In the new coordinate system, the coordinates of the circle are:

$$\text{Circle} = \begin{pmatrix} x'_C \\ y'_C \\ z'_C \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \sin H \\ \cos H \\ 0 \end{pmatrix} \frac{g}{\tan\delta} = \begin{pmatrix} \sin H \\ \cos H\cos\phi \\ -\cos H\sin\phi \end{pmatrix} \frac{g}{\tan\delta}$$

In analysing the simplest case, the extension of a line from the sun to the (appropriate) tip of the gnomon intersected the x-y plane at some point on the circle. This line now intersects the new x'-y' plane too and the point of intersection indicates where the shadow of the tip lies on the new plane.

Taking (x'_T, y'_T, z'_T) and (x'_C, y'_C, z'_C) as two points, the line between them intersects the $z' = 0$ plane at:

$$x'_S = \frac{x'_C z'_T - x'_T z'_C}{z'_T - z'_C} \qquad y'_S = \frac{y'_C z'_T - y'_T z'_C}{z'_T - z'_C}$$

The subscript S stands for shadow. As previously noted, the line from the (appropriate) tip of the gnomon to the equatorial plane sweeps out a cone and this cone intersects the plane in a circle. The cone intersects a plane with arbitrary orientation in an ellipse or hyperbola and the coordinates (x'_S, y'_S) will in general describe an ellipse or a hyperbola.

The angle θ made by the line from the origin to this point relative to the reference direction of the positive y'-axis is given by:

$$\tan\theta = \frac{x'_S}{y'_S} = \frac{x'_C z'_T - x'_T z'_C}{y'_C z'_T - y'_T z'_C} \tag{9}$$

This derivation involves cancelling the common factor $z'_T - z'_C$. The value of this term is:

$$z'_T - z'_C = \left(\cos\phi + \frac{\cos H \sin\phi}{\tan\delta}\right) g$$

The sign of this term will be of interest when it comes to using the inverse-tangent function but, for the moment, assume that it is positive.

Using the coordinates of the North tip and the circle derived above gives:

$$\tan\theta = \frac{\sin H\cos\phi - 0\,(-\cos H\sin\phi)}{\cos H\cos\phi\cos\phi - \sin\phi\,(-\cos H\sin\phi)} = \frac{\sin H\cos\phi}{\cos H} \tag{10}$$

Again, θ is independent of g or δ. The analysis has involved the further cancelling of a common factor $g^2/\tan\delta$. Taken together with $z'_T - z'_C$ the net term cancelled is:

$$\left(\cos\phi + \frac{\cos H\sin\phi}{\tan\delta}\right)\frac{g^3}{\tan\delta} \tag{11}$$

The presence of the g^3 ensures that this expression can always be forced to be positive. Simply select whichever tip of the gnomon gives rise to the appropriate sign of g.

Example II — The Inverse-Tangent Function and the Reverse Side

In practice, the designer would use a two-argument inverse tangent function to determine the value of θ at a given value of H, perhaps adapting (10) to:

$$\theta = \mathsf{ATAN2}(\cos H, \sin H\cos\phi) \tag{12}$$

This expression gives the angles for the time lines on a North-facing vertical wall sundial at arbitrary latitude. Unsurprisingly this reduces to (4) when $\phi = 0$.

Given that $\cos(\phi) = \cos(-\phi)$ a North-facing wall sundial at latitude $-52°$ could be marked out identically to a wall sundial at latitude $52°$. There is a practical difference between the two. A North-facing wall at latitude $52°$ catches the sun only in the summer months and only for a few hours early in the morning and late afternoon. A North-facing sundial at latitude $-52°$ would catch the sun every day of the year. In both cases the shadow of the gnomon would advance clockwise.

Of course, either sundial would work as a South-facing sundial when viewed from the reverse side but the shadow would go round anti-clockwise. Also it would be the sundial at latitude 52° which would do better in terms of hours of sunshine.

Example II — On which Side is the Sun?

It is worth looking again at expression (11):

$$\left(\cos\phi + \frac{\cos H \sin\phi}{\tan\delta}\right)\frac{g^3}{\tan\delta}$$

This is the common factor which is forced always to be positive by choosing the appropriate sign for g. The sign of g^3 is the same as the sign of g and hence the required sign of g is the sign of the remainder of the expression which, on slight rearrangement, is:

$$\frac{\sin\delta\cos\phi + \cos H\cos\delta\sin\phi}{\sin\delta\tan\delta} \tag{13}$$

The presence of the $\cos H$ means that, in general, the sign depends on the time of day. In practical terms, one has to use the North end of the gnomon at the times of the day when (13) is positive and the South end at times when (13) is negative. This is a simple consequence of the sun sometimes being on one side of a wall and sometimes being on the other side.

A wall at the equator is a special case. When $\phi = 0$ the term including $\cos H$ vanishes and the sign (13) depends only on the sign of δ. As previously noted, the sun is on the North side of the equatorial plane from roughly 21 March to 23 September and on the South side for the rest of the year.

For a plane at any other orientation the sun is, in general, on one side for part of each day and on the other side for the rest of each day. Nevertheless, if the plane is inclined to the equatorial plane by a small angle the sun can, at appropriate declinations, stay on the same time for several days, or even weeks, at a time.

Consider (13) at 12 noon when $H = 180°$ and $\cos H = -1$:

$$\frac{\sin\delta\cos\phi - \cos\delta\sin\phi}{\sin\delta\tan\delta} = \frac{\sin(\delta-\phi)}{\sin\delta\tan\delta}$$

Given that $-23\frac{1}{2} < \delta < 23\frac{1}{2}$ the signs of $\sin\delta$ and $\tan\delta$ will necessarily be the same so the sign of the expression is the sign of $\sin(\delta-\phi)$.

Now, $-113\frac{1}{2} < \delta-\phi < 113\frac{1}{2}$ and, over this range, $\sin(\delta-\phi)$ has the same sign as $\delta-\phi$. Accordingly, the sun shines on the North side of the plane of the dial at noon if $\delta-\phi > 0$ and on the South side otherwise.

In the northern hemisphere the sun shines on the North side of the plane at noon if $\delta > \phi$ and this can happen only if $\phi < 23\frac{1}{2}^{\circ}$, requiring the latitude to be no further North than the Tropic of Cancer. In such circumstances the sun will stay on the same side of the plane throughout the day.

In the southern hemisphere the sun shines on the South side of the plane at noon if $\delta < \phi$ and this can happen only if $\phi > -23\frac{1}{2}^\circ$, requiring the latitude to be no further South than the Tropic of Capricorn. Again, in such circumstances the sun will stay on the same side of the plane throughout the day.

Example II — Guidelines for the Designer

Assuming that the sundial of Example II really is intended for a vertical North-facing wall at latitude ϕ, here is a summary of the design stages which might now follow:

1. Choose a position on the wall for the root of the gnomon.
2. Use this point as an origin and draw x'- and y'-axes through the point. The positive y'-axis should be vertical. This is the reference direction.
3. Mount the gnomon so that its root is at the origin. Use (7) to determine the direction of the projection (along the positive y'-axis if $\phi > 0$ and along the negative y'-axis if $\phi < 0$). Use (8) to determine the angle to this projection $(90 - |\phi|)$.
4. Decide on a frame or outline for the dial and for each required time line note the hour-angle H and use (12) to determine the value of θ. Mark the time line along the line from the root of the gnomon at an angle θ to the reference direction.

Example III — Arbitrary Azimuth

Suppose the wall is kept at latitude ϕ but is rotated so that a normal to the North face has azimuth A. The wall is still vertical but it can be anywhere on the Earth's surface and face in any direction.

As with Example II, the required analysis is most conveniently undertaken if one imagines translating the plane of the dial to the equator. One can then consider the original simplest case and determine what happens if the plane of its dial is first rotated about the x-axis by an angle ϕ and then rotated about the y'-axis by an angle A.

The plane of the original dial is coincident with the equatorial plane and its obverse side is deemed to be the North side. The plane can now be orientated via two rotations (by ϕ and by A) so that the obverse side faces in an arbitrary direction.

Note that neither of the two rotations makes any difference to the orientation of the gnomon, which is deemed fixed parallel to the axis of the Earth with its mid-point on the equator. It is the plane of the dial which is reorientated.

The rotations necessarily make a difference to the angle the gnomon makes to the plane and there is potentially a difference in kind between the rotation by ϕ and the rotation by A.

Rotation by ϕ cannot exceed 90° in either direction so, although the gnomon may point steeply up or steeply down, rotation by ϕ alone guarantees that the North end of the gnomon is appropriate since it is on the obverse (North) side of the plane of the dial.

Rotation by A *can* exceed 90° leading to the possibility of the obverse side facing South. In consequence, for arbitrary orientation, the designer cannot assume that the North end of the gnomon is to be used. When specifying the orientation of the gnomon relative to the dial it will be important to select the correct end of the gnomon. More care than hitherto will be required when cancelling common factors.

Consider the vertical North-facing wall at the equator and imagine looking down on it from above as the azimuth of the wall is steadily increased from 0° to 360°. This rotation by a full circle will involve the plane of the dial being coincident with the line of the gnomon when $A = 90°$ and when $A = 270°$. Accordingly, on two occasions in the course of the rotation, the plane of the dial is cut by the line of the gnomon.

For $A < 90°$, the obverse side of the plane of the dial faces within 90° of due North and the North end of the gnomon will be appropriate.

When $A = 90°$, the obverse side of the dial faces due East and the line of the gnomon will lie in the plane of the dial. For $90° < A < 270°$ the obverse side of the dial will face within 90° of South and the South end of the gnomon will be appropriate.

Example III — The Gnomon

Call the system of coordinates for the new dial (x'', y'', z'') and arrange for the origin to be coincident with the origin used in both Example I and Example II.

The x''-y'' plane is the plane of the dial and, viewed from the obverse side, the x''-axis runs horizontally from left to right. The x''- and z''-axes make an angle A with the x'- and z'-axes respectively. (The x'- and z'-axes are coincident with the original x- and z-axes respectively.) The y''-axis runs upwards and is coincident with the y'-axis.

Any point in the new coordinate system (x'', y'', z'') is related to a point (x', y', z') in the previous system by a rotation matrix as follows:

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

The gnomon of the latest dial is still the original gnomon but in the new coordinate system the coordinates of the North tip of the gnomon are:

$$\text{Tip} = \begin{pmatrix} x''_T \\ y''_T \\ z''_T \end{pmatrix} = \begin{pmatrix} \cos\phi \sin A \\ \sin\phi \\ \cos\phi \cos A \end{pmatrix} g$$

By definition, the positive z''-axis is an outward normal of the obverse side of the dial. If the z''_T coordinate of the North tip, $g \cos\phi \cos A$, is positive then the North end of the gnomon is appropriate for the obverse side of the dial. If the z''_T coordinate is negative then the South end of the gnomon is appropriate.

Given that the sign of g is positive and given that $-90° < \phi < +90°$, $\cos\phi > 0$, the sign of the z''_T coordinate is determined by the sign of $\cos A$.

Using the new coordinates of the North tip, the orientation of the gnomon relative to the plane of the dial can be specified in terms of the angles *proj* and *ang* by trivially adapting the expressions given in (6):

$$\tan(proj) = \frac{x''_T}{y''_T} = \frac{g\cos\phi\sin A}{g\sin\phi} \qquad \text{and} \qquad \sin(ang) = \frac{z''_T}{g} = \cos\phi\cos A$$

The expression for $\tan(proj)$ is fine if the azimuth is within 90° of due North. For example, if $\phi = A = 45°$ the numerator and denominator are both positive so the projection of the North end of the gnomon falls in the quadrant of the x''-y'' plane where both x'' and y'' are positive. To someone looking at the obverse side the projection falls in the upper right-hand quadrant. On the reverse side the same projection would appear in the upper left-hand quadrant.

If ϕ is held at 45° but the azimuth is increased by 180° so that $\phi = 45°$ and $A = 225°$ the obverse side of the dial becomes what was the reverse side. The numerator of the expression for $\tan(proj)$ is now negative, the denominator is still positive so to someone looking at what is now the obverse side the projection of the North end of the gnomon appears in the upper left-hand quadrant.

The result is consistent with that obtained when this was the reverse side but now that this is the obverse side the North end of the gnomon is the wrong end to use. For a shadow to be cast on the obverse side now, the South end of the gnomon should be used and its projection lies in the lower right-hand quadrant.

It is tedious to have two expressions for $\tan(proj)$, one for when the obverse side faces within 90° of due North and another for when it faces within 90° of due South. Happily this potential difficulty can be eliminated by multiplying both the top and bottom lines of the expression for $\tan(proj)$ by z'' coordinate of the North tip:

$$\tan(proj) = \frac{g\cos\phi\sin A}{g\sin\phi} \times \frac{g\cos\phi\cos A}{g\cos\phi\cos A} = \frac{\cos\phi\sin A\cos A}{\sin\phi\cos A}$$

The common factor g^2 can safely be cancelled as can the common factor $\cos\phi$ which is guaranteed to be positive. Using the two-argument inverse-tangent function ATAN2, an appropriate expression for *proj* is:

$$proj = \mathsf{ATAN2}(\sin\phi\cos A, \cos\phi\sin A\cos A) \tag{14}$$

When $\phi = A = 45°$, both arguments are positive and the projection is in the upper right-hand quadrant. When $\phi = 45°$ and $A = 225°$, the numerator (the second argument) is positive but the denominator is negative and the projection is in the lower right-hand quadrant as required.

Note that (14) gives the same results as (7) when $A = 0°$.

The expression for $\sin(ang)$ can readily be used in an inverse-sine function such as ASIN to give an appropriate expression for *ang*

$$ang = \mathsf{ASIN}(\cos\phi\,|\cos A|) \tag{15}$$

Practical considerations demand that the angle of a real gnomon to its projection is such that $0° < ang \leqslant 90°$ which is readily achieved by ensuring that the argument of ASIN is positive. This justifies using $|\cos A|$.

Note that (15) gives the same results as (8) when $A = 0°$.

Example III — The Dial Markings

In the new coordinate system the coordinates of the circle are:

$$\text{Circle} = \begin{pmatrix} x''_C \\ y''_C \\ z''_C \end{pmatrix} = \begin{pmatrix} \sin H \cos A - \cos H \sin\phi \sin A \\ \cos H \cos\phi \\ -\sin H \sin A - \cos H \sin\phi \cos A \end{pmatrix} \frac{g}{\tan\delta}$$

In analysing the simplest case, the extension of a line from the sun to the (appropriate) tip of the gnomon intersected the x-y plane at some point on the circle. In analysing Example II it was noted that this line intersected the x'-y' plane too. It also intersects the x''-y'' plane and the point of intersection indicates where the shadow of the tip lies on the latest plane.

Expression (9) can readily be adapted to the new circumstances. Simply change the single dashes to double dashes throughout. Using the coordinates of the North tip and the circle already derived, gives:

$$\tan\theta = \frac{x''_S}{y''_S} = \frac{x''_C z''_T - x''_T z''_C}{y''_C z''_T - y''_T z''_C}$$

$$= \frac{(\sin H \cos A - \cos H \sin\phi \sin A)\cos\phi \cos A - \cos\phi \sin A\,(-\sin H \sin A - \cos H \sin\phi \cos A)}{\cos H \cos\phi \cos\phi \cos A - \sin\phi\,(-\sin H \sin A - \cos H \sin\phi \cos A)}$$

$$= \frac{\sin H \cos\phi \cos^2 A - \cos H \sin\phi \cos\phi \sin A \cos A + \sin H \cos\phi \sin^2 A + \cos H \sin\phi \cos\phi \sin A \cos A}{\cos H \cos^2\phi \cos A + \sin H \sin\phi \sin A + \cos H \sin^2\phi \cos A)}$$

$$= \frac{\sin H \cos\phi}{\cos H \cos A + \sin H \sin\phi \sin A}$$

Once again the analysis has involved cancelling a common factor which can always be forced to be positive, leaving θ independent of g or δ. The angle θ gives the direction of the shadow of the North end of the gnomon which is appropriate when the azimuth is within 90° of due North. If $90° < A < 270°$ then the South end should be used and the signs of both numerator and denominator should be reversed. As when considering the projection of the gnomon, this is achieved by multiplying both by $\cos A$ giving:

$$\tan\theta = \frac{\sin H \cos\phi \cos A}{\cos H \cos^2 A + \sin H \sin\phi \sin A \cos A} \tag{16}$$

Example III — The Inverse-Tangent Function and the Reverse Side

In practice, the designer would use a two-argument inverse tangent function to determine the value of θ at a given value of H, perhaps adapting (16) to:

$$\theta = \mathsf{ATAN2}(\cos H \cos^2 A + \sin H \sin\phi \sin A \cos A, \sin H \cos\phi \cos A) \tag{17}$$

This expression gives the angles for the time lines on a vertical wall sundial at arbitrary latitude when the wall has arbitrary azimuth. When $A = 0°$ this reduces to (12) and when $\phi = A = 0°$ this reduces to (4).

When $\phi = 0°$ and $A = 180°$ (17) reduces to:

$$\theta = \mathsf{ATAN2}(\cos H, -\sin H)$$

This marks a return to the simplest case of a vertical wall sundial lying in an East-West plane on the equator but taking the South side as the obverse side. This expression for θ was first given as (5) and leads to $\theta = -H$, correctly reflecting the fact that the shadow goes round anti-clockwise.

Example III — Guidelines for the Designer

Assuming that the sundial of Example III really is intended for a vertical wall at latitude ϕ with azimuth A, here is a summary of the design stages which might now follow:

1. Choose a position on the wall for the root of the gnomon.
2. Use this point as an origin and draw x''- and y''-axes through the point. The positive y''-axis should be vertical. This is the reference direction.
3. Mount the gnomon so that its root is at the origin. Use (14) to determine the direction of the projection and use (15) to determine the angle to this projection.
4. Decide on a frame or outline for the dial and for each required time line note the hour-angle H and use (17) to determine the value of θ. Mark the time line along the line from the root of the gnomon at an angle θ to the reference direction.

Example IV — Arbitrary Lean

Suppose the wall is kept at latitude ϕ and the normal to the North face continues to have Azimuth A but the wall now leans over at an angle ℓ in the direction of the normal to the obverse face (what was originally the North face of the wall).

The required analysis is again most conveniently undertaken if one imagines translating the plane of the dial to the equator. One can then consider the original simplest case and determine what happens if the plane of its dial is first rotated about the x-axis by an angle ϕ and then rotated about the y'-axis by an angle A and then rotated about the x''-axis by an angle ℓ.

The plane of the original dial is coincident with the equatorial plane and its obverse side is deemed to be the North side. The plane can now be orientated via three rotations (by ϕ, by A and by ℓ) so that the obverse side is arbitrarily orientated.

Example IV — The Gnomon

Call the system of coordinates for the new dial (x''', y''', z''') and arrange for the origin to be coincident with the origin used in Examples I, II and III.

The x'''-y''' plane is the plane of the dial and, viewed from the obverse side, the x'''-axis runs horizontally from left to right. The x'''-axis is coincident with the x''-axis and the y'''- and z'''-axes make an angle ℓ with the y''- and z''-axes respectively.

Any point in the new coordinate system (x''', y''', z''') is related to a point (x'', y'', z'') in the previous system by a rotation matrix as follows:

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\ell & \sin\ell \\ 0 & -\sin\ell & \cos\ell \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$

The gnomon of the latest dial is still the original gnomon but in the new coordinate system the coordinates of the North tip of the gnomon are:

$$\text{Tip} = \begin{pmatrix} x'''_T \\ y'''_T \\ z'''_T \end{pmatrix} = \begin{pmatrix} \cos\phi\sin A \\ \sin\phi\cos\ell + \cos\phi\cos A\sin\ell \\ -\sin\phi\sin\ell + \cos\phi\cos A\cos\ell \end{pmatrix} g$$

By definition, the positive z'''-axis is an outward normal of the obverse side of the dial. If the z'''_T coordinate of the North tip is positive then the North end of the gnomon is appropriate for the obverse side of the dial. If the z'''_T coordinate is negative then the South end of the gnomon is appropriate. Given that the sign of g is positive, coordinate is determined by the sign of:

$$\cos\phi\cos A\cos\ell - \sin\phi\sin\ell \tag{18}$$

The orientation of the gnomon relative to the plane of the dial can be again be specified in terms of the angles *proj* and *ang*. Using the new coordinates of the North tip and also multiplying both top and bottom lines by the sign of the z''' coordinate to ensure that the numerator and denominator are correctly signed leads to:

$$\tan(proj) = \frac{x'''_T}{y'''_T} = \frac{\cos\phi\sin A(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)}{(\sin\phi\cos\ell + \cos\phi\cos A\sin\ell)(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)}$$

The expression for $\sin(ang)$ is:

$$\sin(ang) = |\cos\phi\cos A\cos\ell - \sin\phi\sin\ell|$$

Hence:

$$\begin{aligned} proj &= \mathsf{ATAN2}((\sin\phi\cos\ell + \cos\phi\cos A\sin\ell)(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell), \\ &\qquad \cos\phi\sin A(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)) \qquad (19) \\ ang &= \mathsf{ASIN}(|\cos\phi\cos A\cos\ell - \sin\phi\sin\ell|) \qquad (20) \end{aligned}$$

Example IV — The Dial Markings

In the new coordinate system the coordinates of the circle are:

$$\text{Circle} = \begin{pmatrix} x'''_C \\ y'''_C \\ z'''_C \end{pmatrix} = \begin{pmatrix} \sin H \cos A - \cos H \sin\phi \sin A \\ \cos H \cos\phi \cos\ell - \sin H \sin A \sin\ell - \cos H \sin\phi \cos A \sin\ell \\ -\cos H \cos\phi \sin\ell - \sin H \sin A \cos\ell - \cos H \sin\phi \cos A \cos\ell \end{pmatrix} \frac{g}{\tan\delta}$$

The extension of a line from the sun to the (appropriate) tip of the gnomon intersects the x-y plane at some point on the circle. This line also intersects the x'-y' plane, the x''-y'' plane and now the x'''-y''' plane too.

Expression (9) can again be adapted to the new circumstances by changing the single dashes to triple dashes throughout. Using the coordinates of the North tip and the circle gives:

$$\tan\theta = \frac{x'''_S}{y'''_S} = \frac{x'''_C z'''_T - x'''_T z'''_C}{y'''_C z'''_T - y'''_T z'''_C}$$

$$= \frac{\begin{array}{l}(\sin H \cos A - \cos H \sin\phi \sin A)(-\sin\phi \sin\ell + \cos\phi \cos A \cos\ell) \\ \quad -(\cos\phi \sin A)(-\cos H \cos\phi \sin\ell - \sin H \sin A \cos\ell - \cos H \sin\phi \cos A \cos\ell)\end{array}}{\begin{array}{l}(\cos H \cos\phi \cos\ell - \sin H \sin A \sin\ell - \cos H \sin\phi \cos A \sin\ell) \\ \quad (-\sin\phi \sin\ell + \cos\phi \cos A \cos\ell) \\ -(\sin\phi \cos\ell + \cos\phi \cos A \sin\ell) \\ \quad (-\cos H \cos\phi \sin\ell - \sin H \sin A \cos\ell - \cos H \sin\phi \cos A \cos\ell)\end{array}}$$

$$= \frac{\begin{array}{l}-\sin H \sin\phi \cos A \sin\ell + \sin H \cos\phi \cos^2 A \cos\ell + \cos H \sin^2\phi \sin A \sin\ell \\ \quad -\cos H \sin\phi \cos\phi \sin A \cos A \cos\ell + \cos H \cos^2\phi \sin A \sin\ell \\ \quad +\sin H \cos\phi \sin^2 A \cos\ell + \cos H \sin\phi \cos\phi \sin A \cos A \cos\ell\end{array}}{\begin{array}{l}-\cos H \sin\phi \cos\phi \sin\ell \cos\ell + \cos H \cos^2\phi \cos A \cos^2\ell \\ \quad +\sin H \sin\phi \sin A \sin^2\ell - \sin H \cos\phi \sin A \cos A \sin\ell \cos\ell \\ \quad +\cos H \sin^2\phi \cos A \sin^2\ell - \cos H \sin\phi \cos\phi \cos^2 A \sin\ell \cos\ell \\ \quad +\cos H \sin\phi \cos\phi \sin\ell \cos\ell + \sin H \sin\phi \sin A \cos^2\ell \\ \quad +\cos H \sin^2\phi \cos A \cos^2\ell + \cos H \cos^2\phi \cos A \sin^2\ell \\ \quad +\sin H \cos\phi \sin A \cos A \sin\ell \cos\ell + \cos H \sin\phi \cos\phi \cos^2 A \sin\ell \cos\ell\end{array}}$$

$$= \frac{\sin H \cos\phi \cos\ell - \sin H \sin\phi \cos A \sin\ell + \cos H \sin A \sin\ell}{\cos H \cos A \cos^2\ell + \cos H \cos A \sin^2\ell + \sin H \sin\phi \sin A}$$

$$= \frac{\sin H \cos\phi \cos\ell - \sin H \sin\phi \cos A \sin\ell + \cos H \sin A \sin\ell}{\cos H \cos A + \sin H \sin\phi \sin A}$$

The analysis again involves cancelling a common factor which can always be forced to be positive, leaving θ independent of g or δ. The angle θ gives the direction of the shadow of the North end of the gnomon which is appropriate when the z'''-axis, as a normal to the obverse side, makes an angle which is less than 90° to the North end of the gnomon.

Otherwise the South end should be used and the signs of both numerator and denominator should be reversed. As when considering the projection of the gnomon, this is achieved by multiply both by (18) giving:

$$\tan\theta = \frac{(\sin H\cos\phi\cos\ell - \sin H\sin\phi\cos A\sin\ell + \cos H\sin A\sin\ell)(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)}{(\cos H\cos A + \sin H\sin\phi\sin A)(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)} \tag{21}$$

It is assumed that the common factor (18) is non-zero. If (18) *is* zero then the gnomon is in the plane of the dial and (21) is invalid.

The numerator and denominator of (21) are appropriate for use in a two-argument inverse-tangent function. Notice that when $\ell = 0°$ (21) reduces to (16).

In general, the shadow does not align with the reference direction (the positive y'''-axis) when $H = 0°$. At this hour-angle:

$$\tan\theta = \frac{\sin A\sin\ell(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)}{\cos A(\cos\phi\cos A\cos\ell - \sin\phi\sin\ell)}$$

To satisfy $\theta = 0°$ when $H = 0°$ requires $\sin A\sin\ell = 0$ (and the denominator must be positive). In particular, either the wall must be vertical (when $\ell = 0°$) or the azimuth must be 0° or 180° (when $\sin A = 0$).

Example IV — Guidelines for the Designer

Assuming that the sundial of Example IV really is intended for a wall at latitude ϕ with azimuth A and lean ℓ, here is a summary of the design stages which might now follow:

1. Choose a position on the wall for the root of the gnomon.
2. Use this point as an origin and draw x'''- and y'''-axes through the point. The x'''-axis should be horizontal and the positive y'''-axis (at right-angles to the x'''-axis) should be a line of greatest slope in the x'''-y''' plane. This is the reference direction.
3. Mount the gnomon so that its root is at the origin. Use (19) to determine the direction of the projection and use (20) to determine the angle to this projection.
4. Decide on a frame or outline for the dial and for each required time line note the hour-angle H and use (21) to determine the value of θ. Mark the time line along the line from the root of the gnomon at an angle θ to the reference direction.

Example V — Horizontal Dials

A sundial at latitude ϕ whose dial is horizontal can be regarded as a wall sundial with $A = 0°$ and $\ell = -90°$ when (21) reduces to:

$$\tan\theta = \frac{\sin H\sin^2\phi}{\cos H\sin\phi}$$

Frank H. King
26 April 1998