Notes on Spherical Triangles

In order to undertake calculations on the celestial sphere, whether for the purposes of astronomy, navigation or designing sundials, some understanding of spherical triangles is essential.

The commonly used formulae for spherical triangles are most readily derived using vector notation. Accordingly, these notes begin with a discussion of vectors.

Vector Identities

Three vector identities are particularly useful. These will be exploited when deriving the formulae used for spherical triangles.

Identity I

Given three arbitrary vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$:

$$(\mathbf{x} \times \mathbf{y}).\mathbf{z} = (\mathbf{y} \times \mathbf{z}).\mathbf{x} = (\mathbf{z} \times \mathbf{x}).\mathbf{y}$$

Proof: each expression is the volume of the same parallelepiped.

Identity II

Given three arbitrary vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$:

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x}.\mathbf{z})\,\mathbf{y} - (\mathbf{x}.\mathbf{y})\,\mathbf{z}$$

Proof: take three mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ arranged so that $\mathbf{y}$ and $\mathbf{j}$ are aligned and that $\mathbf{z}$ is in the plane defined by $\mathbf{j}$ and $\mathbf{k}$. The third vector, $\mathbf{x}$, is arbitrarily aligned. The following figure shows the scheme:

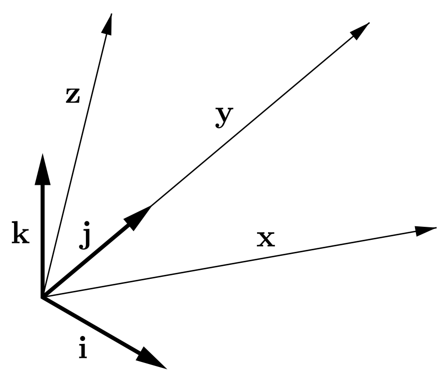

Fig. 1 — Three Arbitrary Vectors

Given the arrangement shown in Fig. 1, scalar values x_1, x_2, x_3, y_2, z_2 and z_3 can be found such that:

$$\begin{aligned}\mathbf{x} &= x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k}\\ \mathbf{y} &= \phantom{x_1\,\mathbf{i} + {}} y_2\,\mathbf{j}\\ \mathbf{z} &= \phantom{x_1\,\mathbf{i} + {}} z_2\,\mathbf{j} + z_3\,\mathbf{k}\end{aligned}$$

Noting that $\mathbf{j}\times\mathbf{j} = 0$ and that $\mathbf{j}\times\mathbf{k} = \mathbf{i}$:

$$\mathbf{y}\times\mathbf{z} = y_2 z_3\,\mathbf{i}$$

Hence:

$$\begin{aligned}\mathbf{x}\times(\mathbf{y}\times\mathbf{z}) &= -x_2 y_2 z_3\,\mathbf{k} + x_3 y_2 z_3\,\mathbf{j}\\ &= x_3 y_2 z_3\,\mathbf{j} - x_2 y_2(-z_2\,\mathbf{j} + z_2\,\mathbf{j} + z_3\,\mathbf{k})\\ &= x_3 z_3 y_2\,\mathbf{j} - x_2 y_2(-z_2\,\mathbf{j} + \mathbf{z})\\ &= (x_2 z_2 + x_3 z_3) y_2\,\mathbf{j} - x_2 y_2\,\mathbf{z}\\ &= (\mathbf{x}.\mathbf{z})\,\mathbf{y} - (\mathbf{x}.\mathbf{y})\,\mathbf{z}\end{aligned}$$

Identity III

Given four arbitrary vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$:

$$(\mathbf{p}\times\mathbf{q}).(\mathbf{r}\times\mathbf{s}) = (\mathbf{p}.\mathbf{r})(\mathbf{q}.\mathbf{s}) - (\mathbf{p}.\mathbf{s})(\mathbf{q}.\mathbf{r})$$

Proof:

$$(\mathbf{p}\times\mathbf{q}).(\mathbf{r}\times\mathbf{s}) = \big(\mathbf{q}\times(\mathbf{r}\times\mathbf{s})\big).\mathbf{p} \qquad \ldots\text{by Identity I}$$

Then, applying Identity II to the right-hand side:

$$\begin{aligned}(\mathbf{p}\times\mathbf{q}).(\mathbf{r}\times\mathbf{s}) &= \big((\mathbf{q}.\mathbf{s})\,\mathbf{r} - (\mathbf{q}.\mathbf{r})\,\mathbf{s}\big).\mathbf{p}\\ &= (\mathbf{p}.\mathbf{r})(\mathbf{q}.\mathbf{s}) - (\mathbf{p}.\mathbf{s})(\mathbf{q}.\mathbf{r})\end{aligned}$$

Spherical Triangles

A spherical triangle is a region on the surface of a sphere bounded by the arcs of three great circles. Without loss of generality, the sphere can be deemed to have unit radius. A typical spherical triangle is shown in Fig. 2:

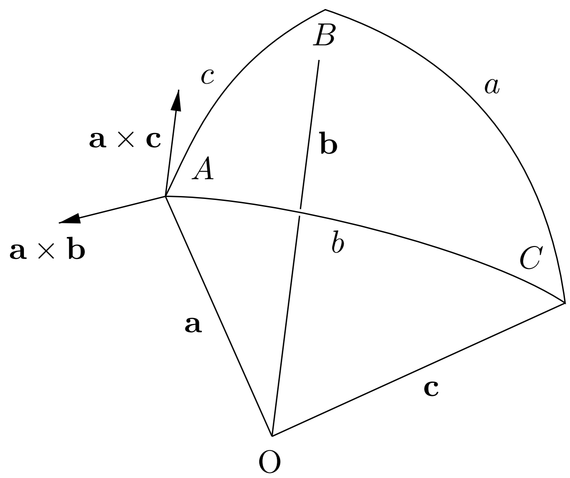

Fig. 2 — A Spherical Triangle

The centre of the sphere is shown as point O. The lengths of the three arcs bounding the triangle are shown as a, b and c. These lengths are measured as angles subtended by the arcs at the centre of the sphere.

The three values A, B and C are the angles at the vertices of the spherical triangle. Conventionally the side whose length is a is arranged opposite the vertex whose angle is A and so on.

Three vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, are also shown in the figure. These are the outward vectors from the centre of the sphere to the three vertices. Clearly the length of each of these vectors is the radius of the sphere and, assuming that the sphere has unit radius, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors.

The First Cosine Rule

The first of the formulae for spherical triangles is:

$$\cos a = \cos b \, \cos c + \sin b \, \sin c \, \cos A$$

To derive this, use Identity III to give:

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{a} \times \mathbf{c}) = (\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{c}) - (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{a}) \tag{1}$$

Now consider the following view of vertex A looking directly towards the centre of the sphere along $-\mathbf{a}$:

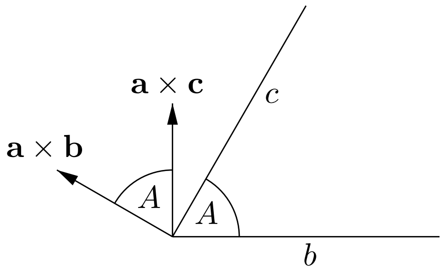

Fig. 3 — Close-up of Vertex A looking along $-\mathbf{a}$

Note that $\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane defined by $\mathbf{a}$ and $\mathbf{b}$ and hence to arc c which is in this plane. Likewise $\mathbf{a} \times \mathbf{c}$ is perpendicular to arc b. Since the angle between arc c and arc b is A, the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ is also A.

Given that c is the angle between $\mathbf{a}$ and $\mathbf{b}$, the magnitude of $\mathbf{a} \times \mathbf{b}$ is $\sin c$. Likewise the magnitude of $\mathbf{a} \times \mathbf{c}$ is $\sin b$. Since the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ is A the left-hand side of (1) is:

$$(\mathbf{a} \times \mathbf{b}).(\mathbf{a} \times \mathbf{c}) = \sin c \cos A \sin b$$

The right-hand side of (1) is trivially:

$$(\mathbf{a}.\mathbf{a})(\mathbf{b}.\mathbf{c}) - (\mathbf{a}.\mathbf{c})(\mathbf{b}.\mathbf{a}) = 1.\cos a - \cos b \cos c$$

Equating the left- and right-hand sides leads directly to the first cosine rule.

A special case is of passing interest. The following figure shows a spherical triangle in which vertex A is approaching 180°:

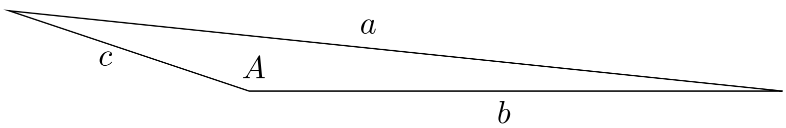

Fig. 4 — Approaching a Degenerate Case

When A=180°, $\cos A = -1$ and the first cosine rule gives:

$$\begin{aligned}\cos a &= \cos b \cos c - \sin b \sin c \\ &= \cos(b + c)\end{aligned}$$

This leads to the obvious result that, in the limit, a=b+c.

The Second Cosine Rule

The second of the formulae for spherical triangles is:

$$-\cos A = \cos B\ \cos C - \sin B\ \sin C\ \cos a$$

To derive this, set up three unit vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ parallel to $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ respectively, as illustrated in the following figure:

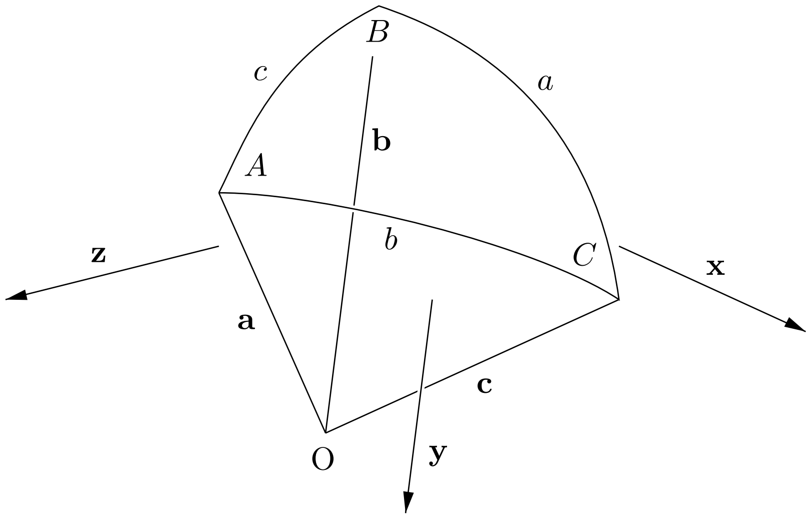

Fig. 5 — A Spherical Triangle

From Identity III:

$$(\mathbf{z} \times \mathbf{x}).(\mathbf{x} \times \mathbf{y}) = (\mathbf{z}.\mathbf{x})(\mathbf{x}.\mathbf{y}) - (\mathbf{z}.\mathbf{y})(\mathbf{x}.\mathbf{x}) \tag{2}$$

Now translate vectors $\mathbf{x}$ and $\mathbf{z}$ to vertex B and consider the following view of that vertex looking directly towards the centre of the sphere along $-\mathbf{b}$:

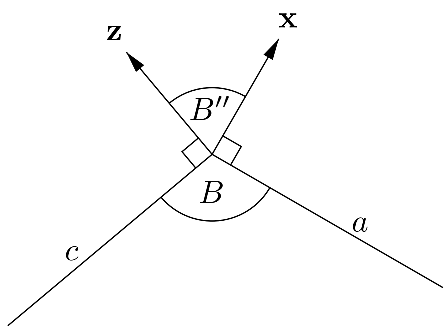

Fig. 6 — Close-up of Vertex B looking along $-\mathbf{b}$

Note that $\mathbf{x}$ is parallel to $\mathbf{b} \times \mathbf{c}$ which is perpendicular to the plane which contains arc a, and hence $\mathbf{x}$ is perpendicular to arc a. Likewise $\mathbf{z}$ is perpendicular to arc c. Since the angle between arc a and arc c is B, it is clear that the angle between $\mathbf{x}$ and $\mathbf{z}$ is $180-B$. This angle is shown as B'' in Fig. 6.

Given that $\mathbf{z}$ and $\mathbf{x}$ are unit vectors, the magnitude of $\mathbf{z} \times \mathbf{x}$ is $\sin B''$ which is the same as $\sin B$. The direction of $\mathbf{z} \times \mathbf{x}$ is *towards* the centre of the sphere along $\mathbf{b}$. Since $\mathbf{b}$ is a unit vector whose direction is *outwards* from the centre of the sphere, $\mathbf{z} \times \mathbf{x} = -\mathbf{b} \sin B$. Likewise $\mathbf{x} \times \mathbf{y} = -\mathbf{c} \sin C$. Accordingly, the left-hand side of (2) is:

$$\begin{aligned}(\mathbf{z} \times \mathbf{x}).(\mathbf{x} \times \mathbf{y}) &= (-\mathbf{b} \sin B).(-\mathbf{c} \sin C) \\ &= \sin B \, \cos a \, \sin C\end{aligned}$$

Given that $\mathbf{z}$ and $\mathbf{x}$ are unit vectors, the magnitude of $\mathbf{z}.\mathbf{x}$ is $\cos B''$ which is the same as $-\cos B$. Likewise $\mathbf{x}.\mathbf{y} = -\cos C$ and $\mathbf{z}.\mathbf{y} = -\cos A$. Accordingly the right-hand side of (2) is:

$$\begin{aligned}(\mathbf{z}.\mathbf{x})(\mathbf{x}.\mathbf{y}) - (\mathbf{z}.\mathbf{y})(\mathbf{x}.\mathbf{x}) &= (-\cos B)(-\cos C) - (-\cos A).1 \\ &= \cos B \, \cos C + \cos A\end{aligned}$$

Equating the left- and right-hand sides leads directly to the second cosine rule.

The Sine Rule

The third of the formulae for spherical triangles is:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

To derive this, use Identity II to give:

$$(\mathbf{a} \times \mathbf{c}) \times (\mathbf{a} \times \mathbf{b}) = \big((\mathbf{a} \times \mathbf{c}).\mathbf{b}\big)\, \mathbf{a} - \big((\mathbf{a} \times \mathbf{c}).\mathbf{a}\big)\, \mathbf{b} \qquad (3)$$

With reference to Fig. 3, the magnitudes of $\mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ have already been shown to be $\sin b$ and $\sin c$ and the angle between them is A. Accordingly the left-hand side of (3) is:

$$(\mathbf{a} \times \mathbf{c}) \times (\mathbf{a} \times \mathbf{b}) = \sin c \, \sin A \, \sin b \, \mathbf{a}$$

The direction of this vector can be verified to be $\mathbf{a}$ by inspection of Fig. 3,

The second term on the right-hand side incorporates $(\mathbf{a} \times \mathbf{c}).\mathbf{a}$ which is zero (vectors $\mathbf{a} \times \mathbf{c}$ and $\mathbf{a}$ are orthogonal so their dot product is zero). Equating the left-hand side to the remaining term on the right-hand side gives:

$$\sin c \, \sin A \, \sin b = (\mathbf{a} \times \mathbf{c}).\mathbf{b} \qquad \text{and similarly} \qquad \sin a \, \sin B \, \sin c = (\mathbf{b} \times \mathbf{a}).\mathbf{c} \qquad (4)$$

By Identity I, the two right-hand sides of (4) are identical, so:

$$\sin c \, \sin A \, \sin b = \sin a \, \sin B \, \sin c$$
$$\sin A \, \sin b = \sin a \, \sin B$$

This leads directly to the first part of the sine rule and, by rotation of identifiers, to the second part.

Note that for small triangles $\sin a \approx a$, $\sin b \approx b$ and $\sin c \approx c$ and the sine rule for spherical triangles is readily seen to be equivalent to the ordinary sine rule for plane triangles.

The First Tangent Rule

The fourth of the formulae for spherical triangles is:

$$\tan a = \frac{\tan A \sin b}{\sin C + \tan A \cos b \cos C}$$

The sides and vertices are labelled as before with a the unknown and A, b and C known.

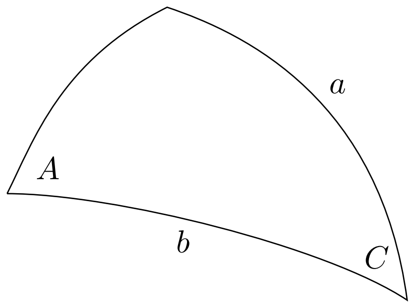

Fig. 7 — A Spherical Triangle

To derive the first tangent rule, use the sine rule:

$$\sin a = \sin A \, \frac{\sin b}{\sin B}$$

Noting that $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$:

$$\cot^2 a = \frac{\sin^2 B}{\sin^2 A \, \sin^2 b} - 1$$

So:

$$\cot^2 a = \frac{1 - \cos^2 B - \sin^2 A \, \sin^2 b}{\sin^2 A \, \sin^2 b} \tag{5}$$

By the second cosine rule:

$$-\cos B = \cos C \, \cos A - \sin C \, \sin A \, \cos b$$

Use this expression for $\cos B$ to rewrite the top line of the right-hand side of (5):

$$1 - (\cos^2 A \, \cos^2 C - 2 \cos A \, \cos C \, \sin A \, \sin C \, \cos b + \sin^2 A \, \sin^2 C \, \cos^2 b) - \sin^2 A \, \sin^2 b$$

Replace the leading 1 by $\cos^2 A + \sin^2 A$:

$$\begin{aligned} = (\cos^2 A + \sin^2 A) - \cos^2 A \, \cos^2 C + 2 \cos A \, \cos C \, \sin A \, \sin C \, \cos b \\ - \sin^2 A \, \sin^2 b - \sin^2 A \, \sin^2 C \, \cos^2 b \end{aligned}$$

Group together terms which contain $\cos^2 A$ and terms which contain $\sin^2 A$:

$$= \cos^2 A\,(1 - \cos^2 C) + 2\cos A\,\cos C\,\sin A\,\sin C\,\cos b + \sin^2 A\,(1 - \sin^2 b - \sin^2 C\,\cos^2 b)$$

Simplify in stages:

$$\begin{aligned}
&= \cos^2 A\,\sin^2 C + 2\cos A\,\cos C\,\sin A\,\sin C\,\cos b + \sin^2 A\,(\cos^2 b - \sin^2 C\,\cos^2 b) \\
&= \cos^2 A\,\sin^2 C + 2\cos A\,\cos C\,\sin A\,\sin C\,\cos b + \sin^2 A\,(1 - \sin^2 C)\,\cos^2 b \\
&= \cos^2 A\,\sin^2 C + 2\cos A\,\cos C\,\sin A\,\sin C\,\cos b + \sin^2 A\,\cos^2 C\,\cos^2 b \\
&= (\cos A\,\sin C + \sin A\,\cos C\,\cos b)^2
\end{aligned}$$

This is the top line of the right-hand side of (5). Accordingly:

$$\cot a = \frac{\cos A\,\sin C + \sin A\,\cos C\,\cos b}{\sin A\,\sin b}$$

Hence:

$$\tan a = \frac{\sin A\,\sin b}{\cos A\,\sin C + \sin A\,\cos b\,\cos C} \tag{6}$$

$$= \frac{\tan A\,\sin b}{\sin C + \tan A\,\cos b\,\cos C} \tag{7}$$

Special Cases:

If C=90, the first tangent rule reduces to:

$$\tan a = \tan A\,\sin b$$

If, further, the triangle is small, $\tan a \approx a$ and $\sin b \approx b$ which leads to:

$$a = b.\tan A$$

This corresponds to the elementary use of the tangent function with a plane right-angled triangle.

Note that although (7) is the version of the first tangent rule which was originally presented, it is safer to use (6) if there is any danger of A approaching 90° (when $\tan A$ diverges).

If A=90, (6) reduces harmlessly to:

$$\tan a = \frac{\sin b}{\cos C\,\cos b} = \frac{\tan b}{\cos C}$$

If, further, the triangle is small, $\tan a \approx a$ and $\tan b \approx b$ which leads to:

$$a.\cos C = b$$

This corresponds to the elementary use of the cosine function with a plane right-angled triangle.

The Second Tangent Rule

The fifth of the formulae for spherical triangles is:

$$\tan A = \frac{\tan a \sin B}{\sin c - \tan a \cos B \cos c}$$

The sides and vertices are labelled as before with A the unknown and a, B and c known.

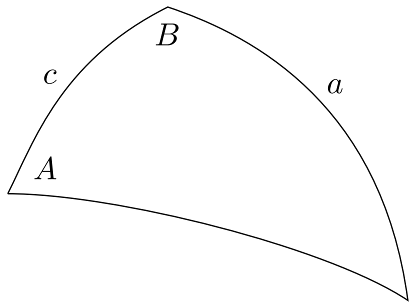

Fig. 8 — A Spherical Triangle

To derive the second tangent rule, use the sine rule:

$$\sin A = \sin a \, \frac{\sin B}{\sin b}$$

Noting that $\cot^2 \theta = \mathrm{cosec}^2 \theta - 1$:

$$\cot^2 A = \frac{\sin^2 b}{\sin^2 a \, \sin^2 B} - 1$$

So:

$$\cot^2 A = \frac{1 - \cos^2 b - \sin^2 a \, \sin^2 B}{\sin^2 a \, \sin^2 B} \tag{8}$$

By the first cosine rule:

$$\cos b = \cos c \, \cos a + \sin c \, \sin a \, \cos B$$

Use this expression for $\cos b$ to rewrite the top line of the right-hand side of (8):

$$1 - (\cos^2 a \, \cos^2 c + 2 \cos a \, \cos c \, \sin a \, \sin c \, \cos B + \sin^2 a \, \sin^2 c \, \cos^2 B) - \sin^2 a \, \sin^2 B$$

Replace the leading 1 by $\cos^2 a + \sin^2 a$:

$$\begin{aligned} &= (\cos^2 a + \sin^2 a) - \cos^2 a \, \cos^2 c - 2 \cos a \, \cos c \, \sin a \, \sin c \, \cos B \\ &\qquad - \sin^2 a \, \sin^2 B - \sin^2 a \, \sin^2 c \, \cos^2 B \end{aligned}$$

Group together terms which contain $\cos^2 a$ and terms which contain $\sin^2 a$:

$$= \cos^2 a\,(1-\cos^2 c) - 2\cos a\,\cos c\,\sin a\,\sin c\,\cos B + \sin^2 a\,(1-\sin^2 B - \sin^2 c\,\cos^2 B)$$

Simplify in stages:

$$\begin{aligned}
&= \cos^2 a\,\sin^2 c - 2\cos a\,\cos c\,\sin a\,\sin c\,\cos B + \sin^2 a\,(\cos^2 B - \sin^2 c\,\cos^2 B) \\
&= \cos^2 a\,\sin^2 c - 2\cos a\,\cos c\,\sin a\,\sin c\,\cos B + \sin^2 a\,(1-\sin^2 c)\,\cos^2 B \\
&= \cos^2 a\,\sin^2 c - 2\cos a\,\cos c\,\sin a\,\sin c\,\cos B + \sin^2 a\,\cos^2 c\,\cos^2 B \\
&= (\cos a\,\sin c - \sin a\,\cos c\,\cos B)^2
\end{aligned}$$

This is the top line of the right-hand side of (8). Accordingly:

$$\cot A = \frac{\cos a\,\sin c - \sin a\,\cos c\,\cos B}{\sin a\,\sin B}$$

Hence:

$$\tan A = \frac{\sin a\,\sin B}{\cos a\,\sin c - \sin a\,\cos B\,\cos c} \qquad (9)$$

$$= \frac{\tan a\,\sin B}{\sin c - \tan a\,\cos B\,\cos c} \qquad (10)$$

Special Cases:

If B=90, the second tangent rule reduces to:

$$\tan A = \frac{\tan a}{\sin c}$$

If, further, the triangle is small, $\tan a \approx a$ and $\sin c \approx c$ which leads to:

$$\tan A = \frac{a}{c}$$

This corresponds to the elementary use of the tangent function with a plane right-angled triangle.
As with the first tangent rule, although (10) is the version which was originally presented, it is safer to use (9) if there is any danger of a approaching 90° (when $\tan a$ diverges).

Frank H. King
30 June 1996