The Shadow of a Point cast onto a Plane

Many sundials incorporate a nodus. Often this is a small ball or disc whose shadow indicates a point on the dial. Sometimes the end of a rod serves as a nodus and, on a large scale, the top of an obelisk can be used in the same way.

This document shows how to determine the position of the shadow of the nodus when the surface of the dial is a plane. The general arrangement is as shown in Fig. 1.

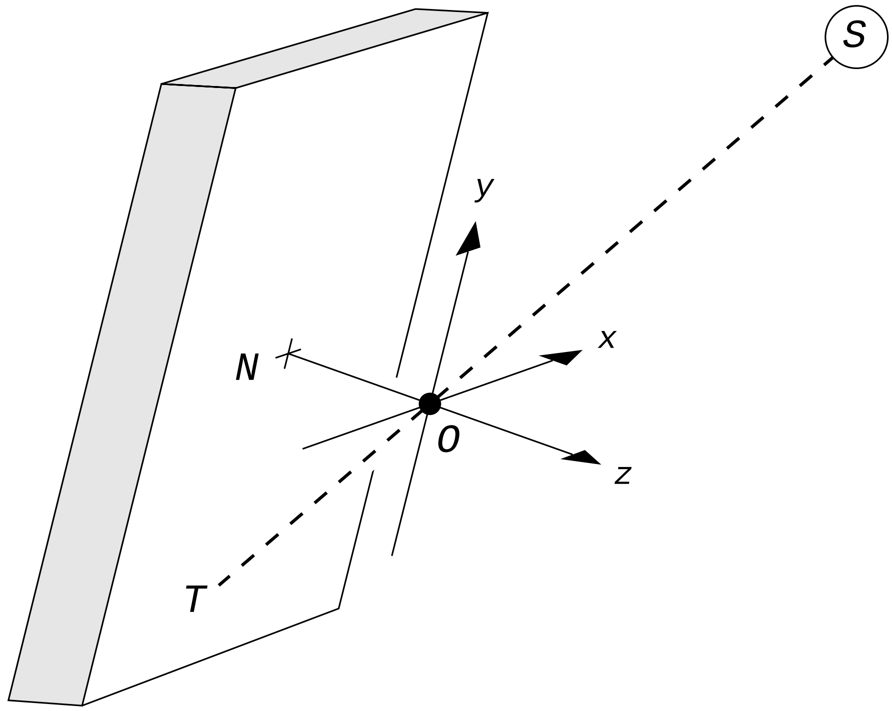

Fig. 1 — The Shadow of a Point cast onto a Plane

The orientation of the plane is arbitrary and the nodus (shown as a solid black circle) is somehow fixed in position at a point, marked O, which is separated from the plane by an appropriate distance.

The foot of the perpendicular from the nodus to the plane is marked N. This is known as the sub-nodus point. The length of the perpendicular ON is known as the nodus height (or ortho-style distance) and will always be taken as n below. Note that $n > 0$.

If the sun, marked S, is on the same side of the plane as the nodus, the shadow of the nodus will fall on the plane at some point. This point is marked T in the figure.

The analysis is strictly geometric. The sun and the nodus are regarded as two points. In general, the straight line between these points will intersect the plane at some third point even if the sun is below the horizon or, even more significantly, on the opposite side of the plane from the nodus. These considerations will be included in the analysis.

It is a practical impossibility for a point to have a shadow but this does not make the exercise any less useful.

Preliminary Considerations

The analysis makes use of a right-handed three-dimensional x, y, z system of rectangular coordinates which is indicated in Fig. 1. In this:

The origin $(0, 0, 0)$ coincides with the nodus at O
The x–y plane is parallel to the plane of the dial
The x-axis is horizontal
The z-axis is perpendicular to the plane of the dial
The direction of increasing z is outwards from the dial

This specification is incomplete if the plane of the dial (and hence the x–y plane) is horizontal. The special case of horizontal dials is considered later.

The z-coordinate of any point on the dial is $-n$ and the coordinates of the sub-nodus point N are $(0, 0, -n)$.

When marking out a design on one side of a wall, it is convenient to use N as the origin and refer to a two-dimensional X, Y system of coordinates where any point (X, Y) coincides with point $(x, y, -n)$. When working in three dimensions, O is unquestionably the more convenient origin which is why the nodus is labelled O.

Given the reference points N and O (and their separation n), the position of the shadow on the plane is governed by:

- The position of the sun
- The orientation of the plane

The position of the sun will be specified by:

$$H = \text{Local Hour Angle } (0^\circ \leqslant H < 360^\circ)$$
$$\delta = \text{Declination } (-23.44^\circ \leqslant \delta \leqslant 24.44^\circ)$$

The hour angle increases by 15° an hour. At local midnight $H = 0^\circ$. At local noon $H = 180^\circ$.

The declination of the sun is the angular distance of the sun above or below the celestial equator. The (approximate) extreme values of the declination are $\pm 23.44^\circ$.

To specify the orientation of the plane, it is convenient to imagine that the dial is marked out on one side of a flat wall whose orientation is fully specified by three parameters:

$$\phi = \text{Latitude of the Wall } (-90^\circ \leqslant \phi \leqslant 90^\circ)$$
$$A = \text{Azimuth of the Outward Normal to the Wall } (0^\circ \leqslant A < 360^\circ)$$
$$\ell = \text{Forward Lean of the Wall } (-90^\circ \leqslant \ell < 90^\circ)$$

Note that these parameters do not specify a unique orientation. A vertical north-facing plane at the equator has the same orientation as a horizontal plane at the north pole.

Note also that the shadow will be at infinity if SO is parallel to the wall.

The Coordinates of the Sun and the Shadow

Using the x, y, z system of coordinates, the coordinates of the sun S and the shadow T will be taken as:

$$S = (S_x, S_y, S_z)$$

$$T = (T_x, T_y, T_z)$$

Given the great distance of the sun from the earth, the absolute values of S_x, S_y and S_z are generally large but, when used in expressions, these coordinates invariably appear as ratios (see (1) below) so large numbers will not feature in calculations.

The absolute values of T_x, T_y and T_z are generally of the same order of magnitude as the nodus height n which is used as a reference measurement. Since the shadow T is in the plane, $T_z = -n$ and is invariant.

Fig. 2 shows Fig. 1 viewed from below. The view is of the z–x plane looking upwards in the direction of the positive y-axis. The sun S and the shadow T are seen as projections onto the z–x plane and the coordinates of these projections are shown as (S_x, S_z) and (T_x, T_z).

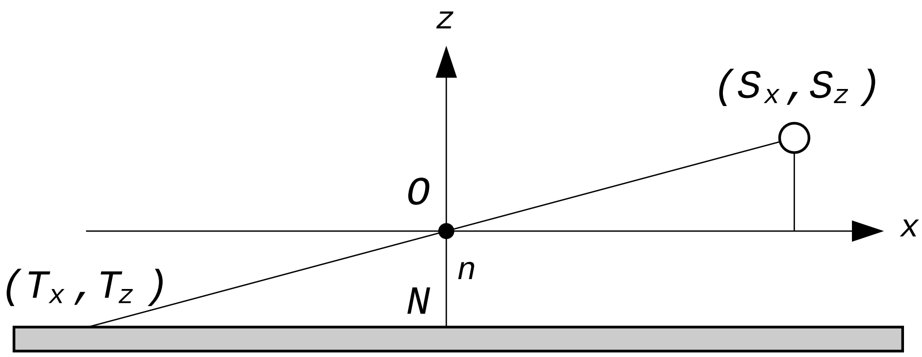

Fig. 2 — Projection of the Sun and the Shadow onto the z–x Plane

By similar triangles, and noting that $T_z = -n$:

$$\frac{S_z}{S_x} = \frac{T_z}{T_x} \qquad \text{or} \qquad T_x = -\frac{S_x}{S_z}\, n$$

The y-coordinate T_y can be determined in the same way and the z-coordinate $T_z = -n$. In summary:

$$T_x = -\frac{S_x}{S_z}\, n \qquad T_y = -\frac{S_y}{S_z}\, n \qquad T_z = -n \tag{1}$$

These expressions for the coordinates of the shadow T apply whatever the orientation of the plane. Accordingly the goal, for an arbitrary orientation of the plane, is to express the coordinates of the sun S in terms of the hour angle H and declination δ.

The goal will be reached in easy stages. The simplest orientation to analyse is that of the equatorial plane. This applies to a north-facing, vertical wall on the equator. The next step is to consider a north-facing wall at arbitrary latitude and then allow arbitrary azimuth. The final step is to give the wall an arbitrary lean as suggested in Fig. 1.

The View from the Centre of the Earth

The large circle in Fig 3 represents the earth as a whole and the small circle represents the distant sun. The straight line from the centre of the earth O to the centre of the sun S intersects the earth's surface at point P. This is the sub-solar point, the point on the earth's surface where the sun is directly overhead. Local sun time at P (and at all other points on the same line of longitude) is 12 noon.

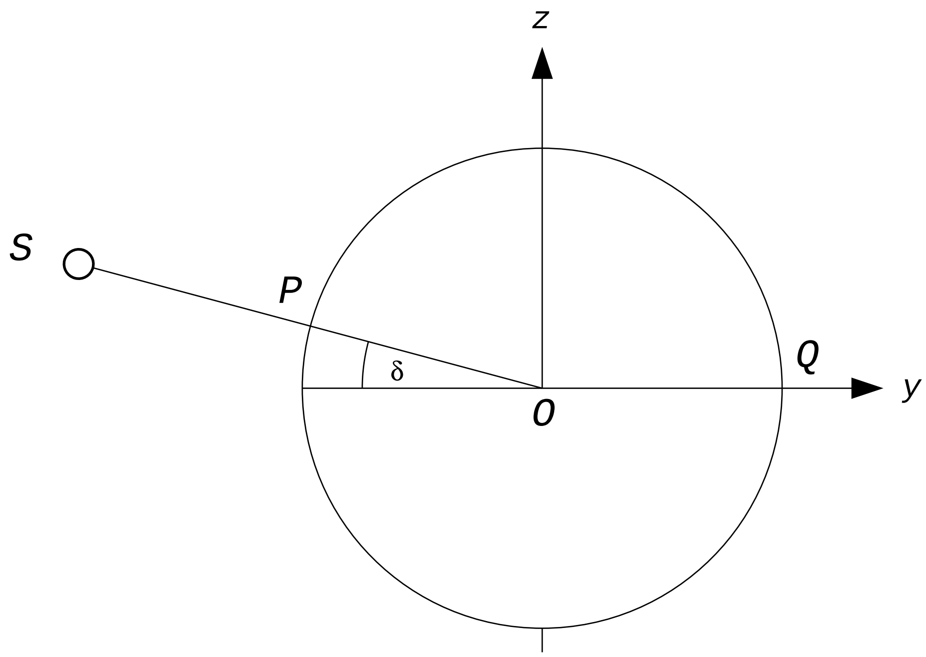

Fig. 3 — Origin at the Centre of the Earth

Point O is the origin of the system of coordinates in which the x–y plane coincides with the equatorial plane and the z-axis coincides with the axis of the earth.

The increasing x-axis (not shown) is out of the plane of the figure. The increasing y-axis intersects the earth's surface at point Q and the increasing z-axis intersects the earth's surface at the north pole.

Note that point Q is on a line of longitude 180° round the earth from P. Local sun time at Q (and at all other points on the same line of longitude) is 12 midnight. The z-axis is perpendicular to the equatorial plane and is coincident with the earth's axis.

All the points O, S, P and Q are in the plane of the figure. The angle made by the line OS to the equatorial plane is δ, the declination of the sun. In the figure δ is about 15°.

If the line OS is regarded as having unit length (an approximation to one astronomical unit) the coordinates of the sun in Fig. 3 are:

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} 0 \\ -\cos\delta \\ \sin\delta \end{pmatrix} \qquad (2)$$

The centre of the sun is in the y–z plane so $S_x = 0$ and the y- and z-coordinates are determined by resolving OS horizontally and vertically in the figure.

The Hour Angle H

If the coordinate system is deemed fixed relative to the earth, the coordinates of the sun will clearly change as the earth rotates.

The rotation is measured by the hour angle H but this angle has to relate to some reference and it will be assumed that $H = 0°$ refers to the state of affairs indicated in Fig. 3. The coordinate S_x is zero and the coordinate S_y is negative.

The time conventionally associated with $H = 0°$ is midnight and this is the time at Q in Fig. 3. Point Q therefore stands on the reference longitude.

The earth rotates from west to east and a few hours after midnight at point Q the position of the sun may be represented as in Fig. 4.

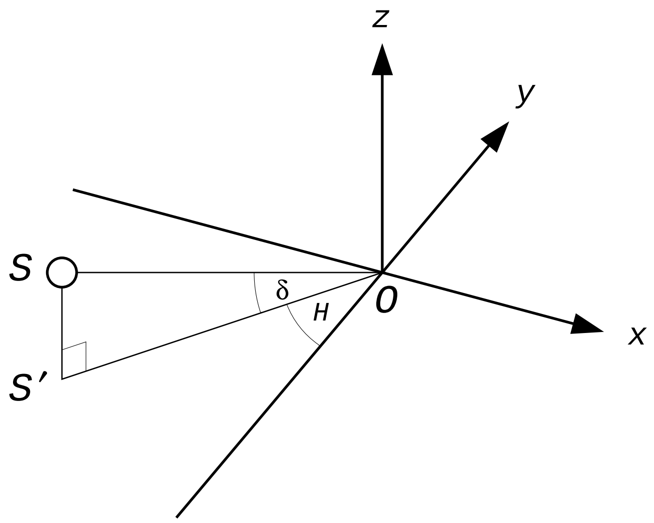

Fig. 4 — The Sun at Hour Angle H

The coordinate system is exactly as in Fig. 3 with point Q (not shown) somewhere on the positive y-axis. In the figure, the hour angle H is a little under 45° (which translates into a time of 3 a.m. at point Q).

The angle made by the line OS to the equatorial plane is again δ but the projection OS' of OS onto the equatorial plane is now at an angle H to the negative y-axis.

Given $OS = 1$, it is clear from the figure that the z-coordinate of the sun is again $\sin\delta$. The x- and y-coordinates are determined by resolving $OS' = \cos\delta$ onto the x- and y-axes. The coordinates of the sun in Fig. 3 are therefore:

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} -\sin H \cos\delta \\ -\cos H \cos\delta \\ \sin\delta \end{pmatrix} \tag{3}$$

Note that when $H = 0°$ these coordinates are as in (2).

Note also that if the hour angle at a given longitude is H, the longitude of the sub-solar point is $180° + H$.

The View from the Surface of the Earth

The above analysis does not change significantly if the origin of the coordinate system is translated from the centre of the earth to a point on the surface, provided that point is on the equator.

Consider Fig. 3 again and imagine constructing a vertical wall at point Q on the equator. Suppose that this wall lies in an east-west plane.

Such an arrangement is shown in Fig. 5 in which the grey rectangle represents the cross-section of the wall. The wall is drawn horizontally but to a local observer it is vertical.

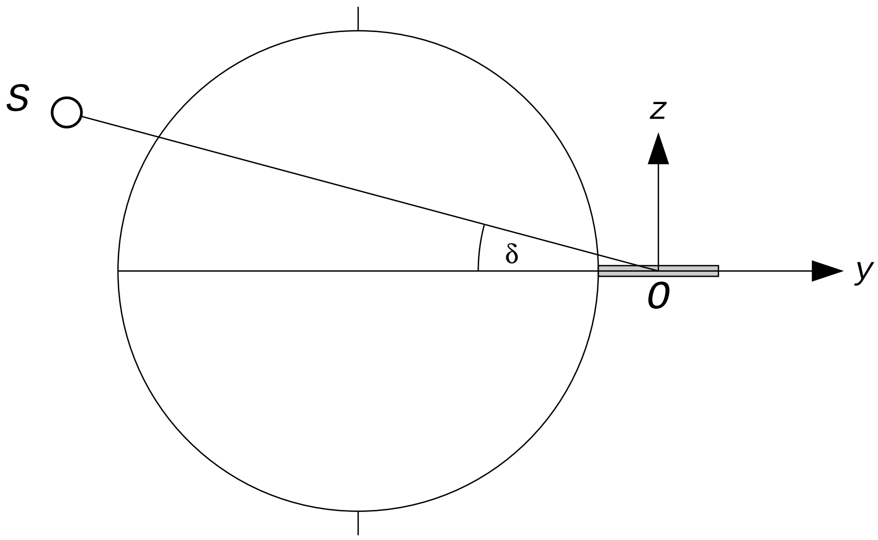

Fig. 5 — A Vertical North-Facing Wall at the Equator

The coordinate system of Fig. 3 has been translated so that its origin O is now in the thickness of the wall as shown.

Once again, the x–y plane coincides with the equatorial plane. The z-axis is parallel to the axis of the earth but no longer coincides with it.

Given the great distance of the sun from the earth, the angle made by the line OS to the x–y plane may be assumed to be δ as before.

To an observer at O the local hour angle $H = 0°$. It is midnight. The sun is clearly below the horizon but its coordinates are again as given in (2).

The local observer may readily consider rotation to be about the (translated) z-axis and at arbitrary hour angle H the coordinates of the sun are given by (3).

Introducing a Nodus

Imagine that the wall in Fig. 5 is displaced a small distance south so that the north face of the wall is separated from point O by a distance n. Label the foot of the perpendicular from O to the north face of the wall N. Arrange for a nodus to be held in place at O.

At some stage during the day, the configuration will be much as in Fig. 2 where the z–x plane is shown rather than the z–y plane.

Case Study I — $\phi = 0,\ A = 0,\ \ell = 0$

The wall just described is appropriate for a vertical north-facing sundial at the equator. The orientation parameters of the wall are:

$$\begin{aligned} \phi &= 0 \quad \text{The wall is at latitude } 0^\circ \\ A &= 0 \quad \text{The azimuth of the outward normal is } 0^\circ \\ \ell &= 0 \quad \text{The wall is vertical so its lean is } 0^\circ \end{aligned}$$

The coordinates of the shadow T are given by (1) and substituting the coordinates of the sun S given by (3) leads to:

$$\text{shadow} = \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} = \begin{pmatrix} -\dfrac{S_x}{S_z} \\ -\dfrac{S_y}{S_z} \\ -1 \end{pmatrix} n = \begin{pmatrix} \dfrac{\sin H \cos\delta}{\sin\delta} \\ \dfrac{\cos H \cos\delta}{\sin\delta} \\ -1 \end{pmatrix} n \tag{4}$$

By inspection, as H increases from 0° to 360° the shadow traces a circle whose centre is at $(0, 0, -n)$ and whose radius is $n/\tan\delta$.

If the wall happens to be square in elevation and if the sub-nodus point N is at the centre of the square then the circle traced might be as shown in Fig. 6.

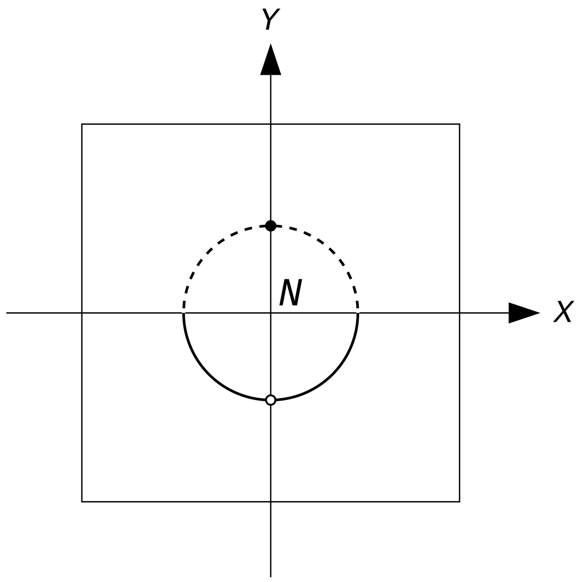

Fig. 6 — A Vertical North-Facing Wall Sundial at the Equator

The upper half of the trace is shown as a broken line. For this portion of the trace, the sun is below the horizon so no shadow would actually be cast. Midnight on the trace is marked by a solid circle and noon is marked by a hollow circle.

The axes shown in Fig. 6 are labelled X and Y to distinguish them from x and y. The X–Y plane is the plane of the dial, parallel to the x–y plane but a distance n from it.

The Horizon Line and the Midnight–Noon Line

The X-axis coincides with the horizon line, the line on the dial which is on the same horizontal level as the nodus. The shadow will fall on this line at sunrise and at sunset when the sun is in the same horizontal plane as the nodus and the horizon line.

The Y-axis coincides with the midnight–noon line, the line on the dial which is in the same vertical north–south plane as the nodus. The shadow will fall on this line at noon when the sun is at its highest point (and the shadow is at its lowest point on the circle).

Constant-Declination Curves

One may imagine that this square dial is marked out on a much larger wall. If printed on A4-paper, Fig. 6 represents a dial 1000 mm square to a scale of 1:20. The nodus height n is assumed to be 100 mm and the circle is appropriate for the summer solstice when $\delta \approx 23.44°$. The radius of the circle is just under 231 mm.

Although the declination is continuously changing, it changes slowly and it is common to mark out on sundials traces that correspond to particular declinations. These are constant-declination curves. In the special case of a dial which is in the plane of the equator (or parallel to the equatorial plane) the constant-declination curves are circles.

The circle in Fig. 6 is the constant-declination curve of minimum radius corresponding to maximum δ. For smaller values of δ the radius ($n/\tan\delta$) is larger but when the value of δ approaches zero the radius approaches infinity. When δ is negative, the sun is on the other side of the wall and no shadow can be cast on the dial side.

A practical approach to marking out a wall sundial is to begin by establishing the nodus height n and the sub-nodus point N and use N as the working origin. The next step is to draw horizontal and vertical lines through N and use these lines as the X- and Y-axes.

Condition for the Sun to be above the Horizon

No shadow can be cast when the sun is below the horizon. For the sun to be above the horizon, the y-coordinate S_y must be positive. From (3):

$$\text{The Sun is above the Horizon if } S_y > 0 \quad \text{or} \quad -\cos H \cos\delta > 0 \tag{5}$$

Within the range of possible values of δ ($\pm 23.44°$) the factor $\cos\delta$ is always positive. Accordingly the sun is above the horizon when $-\cos H > 0$ which requires H to be in the range 90° to 270°. This range corresponds to 6 a.m. to 6 p.m. On the equator, the sun rises at 6 a.m. and sets at 6 p.m. every day of the year.

Condition for the Sun to be on the Dial Side of the Wall

No shadow can be cast on the dial if the sun is on the wrong side of the wall. For the sun to be on the dial side of the wall, the z-coordinate S_z must be positive. From (3):

$$\text{The Sun is on the right side of the wall if } S_z > 0 \quad \text{or} \quad \sin\delta > 0 \tag{6}$$

This condition is equivalent to requiring $\delta > 0°$. When $\delta < 0°$ the sun shines on the south side of the wall (see Fig. 5).

Case Study II — Arbitrary ϕ, $A=0$, $\ell=0$

The wall in Fig. 5 is on the equator. It is north-facing and vertical. Now, keeping it north-facing and vertical, imagine rebuilding the wall at latitude ϕ. This is equivalent to a right-handed rotation by an angle ϕ about the horizontal x-axis.

Fig. 7 represents the new arrangement. The wall is drawn at an angle ϕ to the horizontal but to a local observer it is vertical. The origin O is again in the thickness of the wall but two coordinates systems are shown sharing this origin.

Subject only to a translation, the x, y, z system of coordinates is exactly as in Fig. 5. The x–y plane is parallel to the equatorial plane and the z-axis is parallel to the axis of the earth.

Although not indicated, the angle made by the line OS to the x–y plane is again δ. Using the x, y, z system of coordinates, the coordinates of the sun are as given in (3).

The x', y', z' system of coordinates corresponds to a right-handed rotation of the x, y, z system about the x-axis by an angle ϕ. The x- and x'-axes coincide.

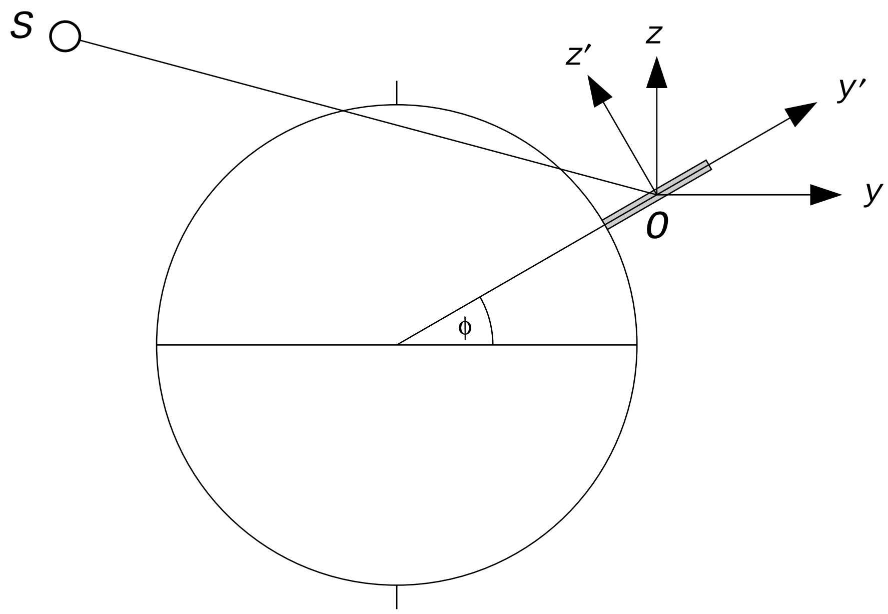

Fig. 7 — A Vertical North-Facing Wall at an Arbitrary Latitude

The coordinates of the sun in the new coordinate system, $(S_{x'}, S_{y'}, S_{z'})$, can readily be determined from the coordinates in the old system, (S_x, S_y, S_z), simply by multiplying by the appropriate rotation matrix:

$$\begin{pmatrix} S_{x'} \\ S_{y'} \\ S_{z'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} =$$

From (3):

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} -\sin H\cos\delta \\ -\cos H\cos\delta \\ \sin\delta \end{pmatrix} = \begin{pmatrix} -\sin H\cos\delta \\ -\cos H\cos\delta\cos\phi + \sin\delta\sin\phi \\ \cos H\cos\delta\sin\phi + \sin\delta\cos\phi \end{pmatrix} \quad (7)$$

Imagine that the wall is again displaced by a small distance along the line of the negative z'-axis so that the perpendicular distance of point O from the wall is n. As before, a nodus is held in place at O and point N is the foot of the perpendicular from O to the wall.

The coordinates of the shadow in the new coordinate system, $(T_{x'}, T_{y'}, T_{z'})$, are given by trivially adapting (1):

$$T_{x'} = -\frac{S_{x'}}{S_{z'}}\, n \qquad T_{y'} = -\frac{S_{y'}}{S_{z'}}\, n \qquad T_{z'} = -n \tag{8}$$

From (8) and using the values of $S_{x'}$, $S_{y'}$ and $S_{z'}$ given in (7):

$$\text{shadow} = \begin{pmatrix} T_{x'} \\ T_{y'} \\ T_{z'} \end{pmatrix} = \begin{pmatrix} -\dfrac{S_{x'}}{S_{z'}} \\ -\dfrac{S_{y'}}{S_{z'}} \\ -1 \end{pmatrix} n = \begin{pmatrix} \dfrac{\sin H \cos\delta}{\cos H \cos\delta \sin\phi + \sin\delta\cos\phi} \\ \dfrac{\cos H \cos\delta \cos\phi - \sin\delta \sin\phi}{\cos H \cos\delta \sin\phi + \sin\delta\cos\phi} \\ -1 \end{pmatrix} n \tag{9}$$

When $\phi = 0$ this reduces to (4). If δ is high and ϕ is small, then as H increases from 0° to 360° the shadow traces an ellipse.

If the 1000 mm square dial in Fig. 6 is rebuilt at latitude $\phi = 8°$ N the ellipse traced when $\delta \approx 23.44°$ would appear as in Fig. 8.

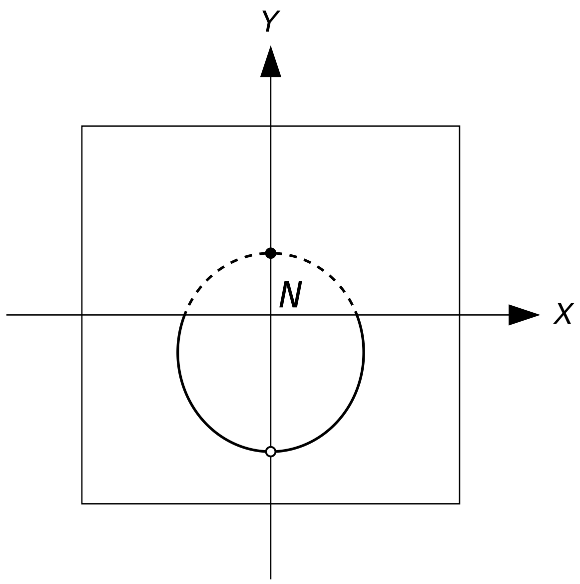

Fig. 8 — A Vertical North-Facing Wall Sundial at 8° *N*

The portion of the trace where the sun is below the horizon is again shown by a broken line. The X-axis again coincides with the horizon line and the Y-axis again coincides with the midnight–noon line.

It is only when absolute value of δ is high and the absolute value of ϕ is very small that the trace is an ellipse. The trace is much more commonly a hyperbola.

Condition for the Sun to be above the Horizon

The condition for the sun to be above the horizon now is that $S_{y'} > 0$. From (7):

$$S_{y'} > 0 \quad \text{if} \quad -\cos H \cos\delta \cos\phi + \sin\delta \sin\phi > 0 \tag{10}$$

This condition reduces to (5) if $\phi = 0°$.

Whether or not the sun is above the horizon does not depend on the orientation of the wall (its azimuth and angle of lean) but only on the hour angle and declination of the sun and the latitude of the observer. Accordingly, the condition for the sun being above the horizon does not become any more ambitious than this.

The condition can be expressed in several ways, one of which is:

$$\tan\delta \tan\phi > \cos H$$

The maximum possible value of $\cos H$ is 1 and if the condition:

$$\tan\delta \tan\phi > 1$$

is satisfied the sun will be above the horizon for all values of H. This occurs in high northern latitudes near the June solstice and in high southern latitudes near the December solstice. The condition reduces to $\phi > 90° - \delta$ and $\phi < -90° - \delta$ respectively.

For example, if $\delta = 10°$ there will be continuous daylight if $\phi > 80°$ and if $\delta = -10°$ there will be continuous daylight if $\phi < -80°$.

Condition for the Sun to be on the Dial Side of the Wall

The condition for the sun to be on the dial side of the wall now is that $S_{z'} > 0$. From (7):

$$S_{z'} > 0 \quad \text{if} \quad \cos H \cos\delta \sin\phi + \sin\delta \cos\phi > 0 \tag{11}$$

This condition reduces to (6) if $\phi = 0°$.

This condition can also be expressed in several ways but, unlike (10), this is not the general case. Whether or not the sun is on the dial side of the wall clearly depends on the orientation of the wall.

Case Study III — Arbitrary ϕ and A, $\ell = 0$

The wall in Fig. 7 is at latitude ϕ. It is north-facing and vertical. Now, keeping the wall vertical and at the same latitude, imagine realigning the wall so that the azimuth of the normal on the dial side is A. This is equivalent to a left-handed rotation by an angle A about the vertical y'-axis.

Fig. 9 represents the new arrangement viewed in plan looking down from above. The origin O is again in the thickness of the wall but two coordinate systems are shown sharing this origin. As in Fig. 7, the z'-axis is horizontal and points north and the x'-axis (not shown in Fig. 7) is horizontal and points west.

The x'', y'', z'' system of coordinates corresponds to a left-handed rotation of the x', y', z' system about the y'-axis by an angle A. The y'- and y''-axes coincide.

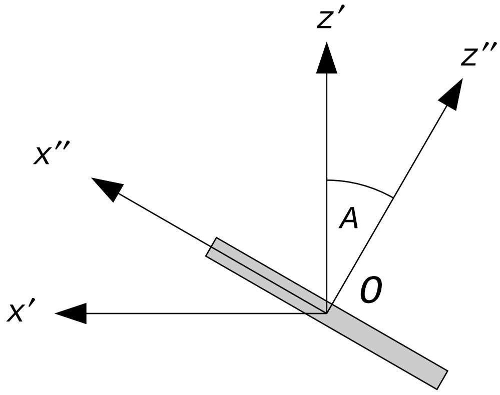

Fig. 9 — A Vertical Wall viewed in plan, Latitude and Azimuth Arbitrary

The coordinates of the sun in the new coordinate system, $(S_{x''}, S_{y''}, S_{z''})$, can readily be determined from the coordinates in the old system, $(S_{x'}, S_{y'}, S_{z'})$, by multiplying by the appropriate rotation matrix:

$$\begin{pmatrix} S_{x''} \\ S_{y''} \\ S_{z''} \end{pmatrix} = \begin{pmatrix} \cos A & 0 & \sin A \\ 0 & 1 & 0 \\ -\sin A & 0 & \cos A \end{pmatrix} \begin{pmatrix} S_{x'} \\ S_{y'} \\ S_{z'} \end{pmatrix}$$

Hence, from (7):

$$\begin{pmatrix} S_{x''} \\ S_{y''} \\ S_{z''} \end{pmatrix} = \begin{pmatrix} -\sin H \cos\delta \cos A + \cos H \cos\delta \sin\phi \sin A + \sin\delta \cos\phi \sin A \\ -\cos H \cos\delta \cos\phi + \sin\delta \sin\phi \\ \sin H \cos\delta \sin A + \cos H \cos\delta \sin\phi \cos A + \sin\delta \cos\phi \cos A \end{pmatrix} \tag{12}$$

Imagine that the wall is again displaced by a small distance along the line of the negative z''-axis so that the perpendicular distance of point O from the wall is n. As before, a nodus is held in place at O and point N is the foot of the perpendicular from O to the wall.

The coordinates of the shadow in the new coordinate system, $(T_{x''}, T_{y''}, T_{z''})$, are given by trivially adapting (1):

$$T_{x''} = -\frac{S_{x''}}{S_{z''}}\, n \qquad T_{y''} = -\frac{S_{y''}}{S_{z''}}\, n \qquad T_{z''} = -n \tag{13}$$

From (13) and using the values of $S_{x''}$, $S_{y''}$ and $S_{z''}$ given in (12):

$$\begin{pmatrix} T_{x''} \\ T_{y''} \\ T_{z''} \end{pmatrix} = \begin{pmatrix} \dfrac{\sin H \cos\delta \cos A - \cos H \cos\delta \sin\phi \sin A - \sin\delta \cos\phi \sin A}{\sin H \cos\delta \sin A + \cos H \cos\delta \sin\phi \cos A + \sin\delta \cos\phi \cos A} \\ \dfrac{\cos H \cos\delta \cos\phi - \sin\delta \sin\phi}{\sin H \cos\delta \sin A + \cos H \cos\delta \sin\phi \cos A + \sin\delta \cos\phi \cos A} \\ -1 \end{pmatrix} n \tag{14}$$

When $A = 0$ this reduces to (9). If δ is high and ϕ and A are small, then as H increases from 0° to 360° the shadow traces an ellipse.

If the 1000 mm square dial in Fig. 8 is rotated so that its azimuth $A = 6°$ the ellipse traced when $\delta \approx 23.44°$ would appear as in Fig. 10.

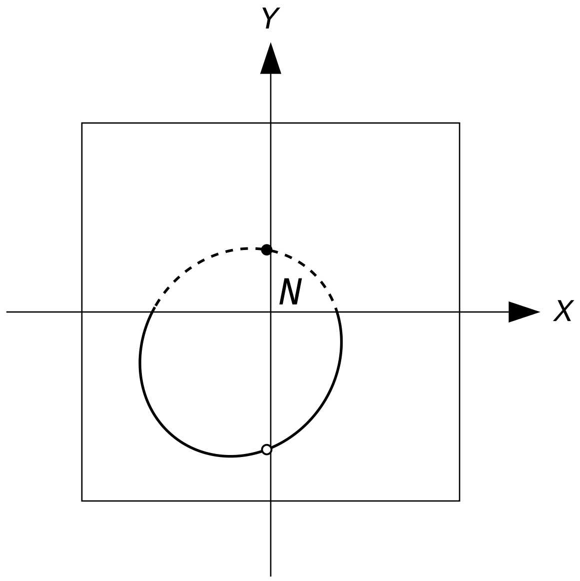

Fig. 10 — A Sundial at 8° N on a Vertical Wall which faces 6° East of North

The portion of the trace where the sun is below the horizon is again shown by a broken line. The X-axis again coincides with the horizon line. The midnight–noon line is vertical as in Fig. 8 but does not coincide with the Y-axis. The midnight and noon points on the trace are displaced slightly in the direction of the negative X-axis.

It is only when absolute value of δ is high and the absolute values of ϕ and A are very small that the trace is an ellipse. The trace is much more commonly a hyperbola.

Condition for the Sun to be on the Dial Side of the Wall

The condition for the sun to be on the dial side of the wall now is that $S_{z''} > 0$. From (12) this condition is that:

$$\sin H \cos\delta \sin A + \cos H \cos\delta \sin\phi \cos A + \sin\delta \cos\phi \cos A > 0 \tag{15}$$

This condition reduces to (11) if $A = 0°$.

Case Study IV — Arbitrary $\boldsymbol{\phi}$, $\boldsymbol{A}$ and $\boldsymbol{\ell}$

The wall in Fig. 9 is at latitude ϕ and the azimuth of the normal on the dial side is A. Now, maintaining ϕ and A, lean the wall over by an angle ℓ. This is equivalent to a right-handed rotation by an angle ℓ about the horizontal x''-axis. Positive ℓ indicates a forward lean of the wall in the direction of the outward normal on the dial side.

Fig. 11 represents the new arrangement viewed in elevation looking in the direction of the negative x''-axis. The origin O is again in the thickness of the wall but two coordinate systems are shown sharing this origin. As in Fig. 9, the z''-axis is horizontal and the y''-axis (not shown in Fig. 9) is vertical.

The x''', y''', z''' system of coordinates corresponds to a right-handed rotation of the x'', y'', z'' system about the x''-axis by an angle ℓ. The x''- and x'''-axes coincide.

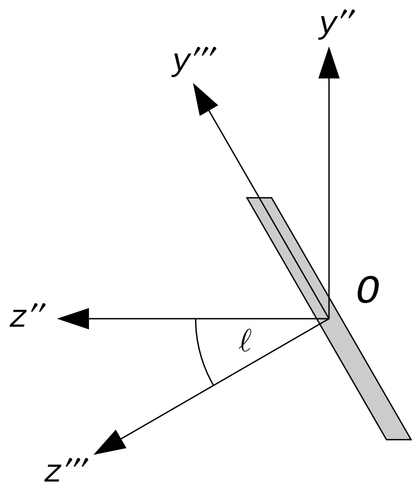

Fig. 11 — A Wall viewed in elevation, Latitude, Azimuth and Lean Arbitrary

The coordinates of the sun in the new coordinate system, $(S_{x'''}, S_{y'''}, S_{z'''})$, can readily be determined from the coordinates in the old system, $(S_{x''}, S_{y''}, S_{z''})$, by multiplying by the appropriate rotation matrix:

$$\begin{pmatrix} S_{x'''} \\ S_{y'''} \\ S_{z'''} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\ell & \sin\ell \\ 0 & -\sin\ell & \cos\ell \end{pmatrix} \begin{pmatrix} S_{x''} \\ S_{y''} \\ S_{z''} \end{pmatrix}$$

Hence, from (12):

$$\begin{pmatrix} S_{x'''} \\ S_{y'''} \\ S_{z'''} \end{pmatrix} = \begin{pmatrix} -\sin H\cos\delta\cos A + \cos H\cos\delta\sin\phi\sin A + \sin\delta\cos\phi\sin A \\ -\cos H\cos\delta\cos\phi\cos\ell + \sin\delta\sin\phi\cos\ell + \sin H\cos\delta\sin A\sin\ell \\ \qquad + \cos H\cos\delta\sin\phi\cos A\sin\ell + \sin\delta\cos\phi\cos A\sin\ell \\ \cos H\cos\delta\cos\phi\sin\ell - \sin\delta\sin\phi\sin\ell + \sin H\cos\delta\sin A\cos\ell \\ \qquad + \cos H\cos\delta\sin\phi\cos A\cos\ell + \sin\delta\cos\phi\cos A\cos\ell \end{pmatrix} \tag{16}$$

Imagine that the wall is again displaced by a small distance along the line of the negative z'''-axis so that the perpendicular distance of point O from the wall is n. As before, a nodus is held in place at O and point N is the foot of the perpendicular from O to the wall. The arrangement now is very like the general case illustrated in Fig. 1.

The coordinates of the shadow in the new coordinate system, $(T_{x'''}, T_{y'''}, T_{z'''})$, are given by trivially adapting (1):

$$T_{x'''} = -\frac{S_{x'''}}{S_{z'''}}\, n \qquad T_{y'''} = -\frac{S_{y'''}}{S_{z'''}}\, n \qquad T_{z'''} = -n \tag{17}$$

From (17) and using the values of $S_{x'''}$, $S_{y'''}$ and $S_{z'''}$ given in (16):

$$\begin{pmatrix} T_{x'''} \\ \\ T_{y'''} \\ \\ T_{z'''} \end{pmatrix} = \begin{pmatrix} \dfrac{\sin H\cos\delta\cos A - \cos H\cos\delta\sin\phi\sin A - \sin\delta\cos\phi\sin A}{\begin{array}{r}\cos H\cos\delta\cos\phi\sin\ell - \sin\delta\sin\phi\sin\ell + \sin H\cos\delta\sin A\cos\ell \\ + \cos H\cos\delta\sin\phi\cos A\cos\ell + \sin\delta\cos\phi\cos A\cos\ell\end{array}} \\ \dfrac{\begin{array}{r}\cos H\cos\delta\cos\phi\cos\ell - \sin\delta\sin\phi\cos\ell - \sin H\cos\delta\sin A\sin\ell \\ - \cos H\cos\delta\sin\phi\cos A\sin\ell - \sin\delta\cos\phi\cos A\sin\ell\end{array}}{\begin{array}{r}\cos H\cos\delta\cos\phi\sin\ell - \sin\delta\sin\phi\sin\ell + \sin H\cos\delta\sin A\cos\ell \\ + \cos H\cos\delta\sin\phi\cos A\cos\ell + \sin\delta\cos\phi\cos A\cos\ell\end{array}} \\ -1 \end{pmatrix} n \quad (18)$$

When $\ell = 0$ this reduces to (14). If δ is high and ϕ, A and ℓ are all small, then as H increases from 0° to 360° the shadow traces an ellipse.

If the 1000 mm square dial in Fig. 10 is rotated so that its lean $\ell = 3°$ the ellipse traced when $\delta \approx 23.44°$ would appear as in Fig. 12.

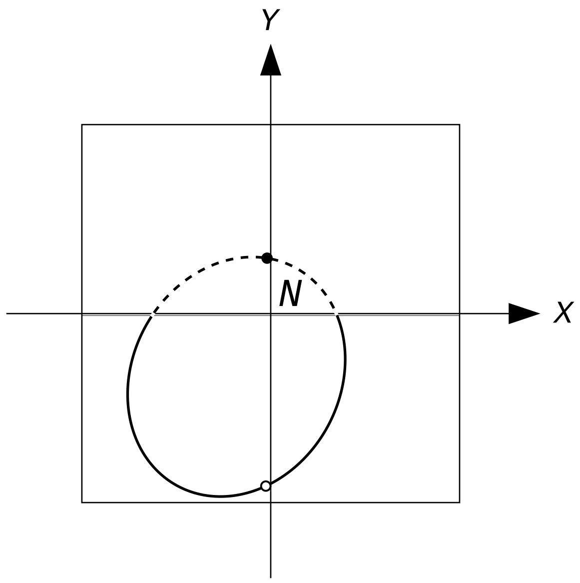

Fig. 12 — A Sundial at 8° N on a Wall which faces 6° East of North and Leans 3°

Note that the Y-axis is parallel to the y'''-axis which is not vertical. Accordingly, unlike the views in Figs 6, 8 and 10, the view in Fig. 12 is not an elevation.

Given a positive value of ℓ the forward lean of the wall means that the nodus at O is at a lower horizontal level than the foot of the perpendicular from the nodus to the dial at N. See Fig. 1.

The horizon line on the dial is at the same horizontal level as the nodus and is therefore parallel to but below the X-axis. The horizon line is shown as a thin line in Fig. 12 but the separation from the X-axis is only $n \tan\ell$ and for a lean of just 3° this separation is almost too small to notice in the figure.

The portion of the trace where the sun is below the horizon is again shown by a broken line and, strictly, this is the portion above the horizon line rather than above the X-axis.

The midnight–noon line is no longer vertical. The midnight point on the trace is displaced slightly in the direction of the negative X-axis and the noon point is displaced somewhat further in the same direction.

It is only when absolute value of δ is high and the absolute values of ϕ, A and ℓ are very small that the trace is an ellipse. The trace is much more commonly a hyperbola.

Condition for the Sun to be on the Dial Side of the Wall

The condition for the sun to be on the dial side of the wall now is that $S_{z'''} > 0$. From (16) this condition is that:

$$\begin{aligned}\cos H \cos\delta \cos\phi \sin\ell - \sin\delta \sin\phi \sin\ell + \sin H \cos\delta \sin A \cos\ell \\ + \cos H \cos\delta \sin\phi \cos A \cos\ell + \sin\delta \cos\phi \cos A \cos\ell > 0\end{aligned} \tag{19}$$

This condition reduces to (15) if $\ell = 0°$.

Using a Total Station — Another Set of Axes

The expressions for $T_{x'''}$ and $T_{y'''}$ in (18) may seem rather daunting but it takes only a few minutes to key them into a spreadsheet or computer program and thereby generate traces like that in Fig. 12. If the dial is to be set out on a real wall and the dimensions are of the order of metres the challenge is altogether greater.

It is convenient to use standard surveying techniques. An important tool used by surveyors is a total station, an instrument which directly employs an x, y, z system of coordinates. The origin may be chosen freely but there is an important constraint on the orientation. The x–z plane has to be horizontal.

A total station is therefore particularly convenient when the wall is vertical. In outline, the surveying procedure for a vertical wall might be as follows:

1. Survey the (surface of the) wall to establish the best-fit vertical plane.
2. Determine the azimuth of the outward normal to the best-fit vertical plane. [A good method is to use equipment that exploits the Global Positioning System (GPS).]
3. Configure the total station so that the centre of the nodus, O, is the origin, the x-axis is horizontal and parallel to the best-fit plane, the y-axis is vertical, and the z-axis is horizontal and normal to the best-fit plane.
4. Establish the point N on the wall which has the same x- and y-coordinates as the centre of the nodus. Use this point as the origin on the wall.
5. Determine the distance ON and call this n.
6. Mark out a grid of 1 m squares on the wall using N as the origin. This grid can be used as giant graph paper when setting out the design.

Step 3 configures the total station to employ the x'', y'', z'' system of coordinates used in the expressions for $T_{x''}$, $T_{y''}$ and $T_{z''}$ given in (14).

If the wall is inclined to the vertical, the procedure has to be modified:

1. Survey the wall to establish the best-fit plane and determine l, its angle of inclination to the vertical.
2. As before.
3. Configure the total station so that the centre of the nodus, O, is the origin, the x-axis is horizontal and parallel to the best-fit plane, the y-axis is vertical, and the z-axis is horizontal (*but not now normal to the best-fit plane*).
4. Establish the point N' on the wall which has the same x- and y-coordinates as the centre of the nodus. Use this point as the origin on the wall.
5. Determine the distance ON' and call this n'. Note that $ON = n = n' \cos \ell$.
6. Mark out a grid of 1 m squares on the wall using N' as the origin.

Fig. 13 shows the wall in elevation. The modified procedure configures the total station to employ a new system of x'''', y'''', z'''' coordinates. The y''''- and z''''-axes are shown in the figure. The y'''- and z'''-axes are as in Fig. 11 but the wall has been displaced along the negative z'''-axis and a nodus introduced at the origin O.

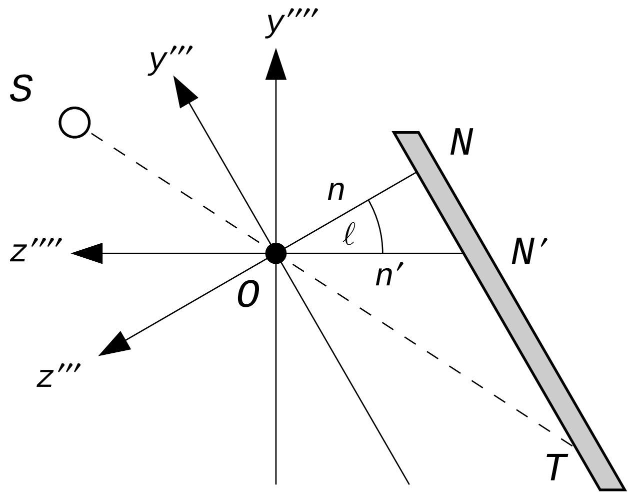

Fig. 13 — A Sundial on a Leaning Wall

Using a total station it is straightforward to establish point N' which is on the same horizontal level as the nodus (and so marks the horizon line) and a surveyor would naturally use this point as the origin on the wall. Point N can readily be established indirectly but, when marking out a grid of 1 m squares, it is best to stick with N' as the origin.

On the wall, the horizontal lines of the grid are at 1 m intervals but, to the total station, these lines are at vertical intervals of $\cos \ell$ metres. The lines of the grid which follow lines of greatest slope on the wall are at horizontal intervals of 1 m.

When setting out the design using this grid, the values of $T_{x'''}$ and $T_{y'''}$ given by (18) should be used but note that $n = n' \cos \ell$ and that an offset of NN' (or $n' \sin \ell$) must be added to each $T_{y'''}$ value to compensate for the shift of origin from N to N'.

Using a Total Station — A Final Rotation

Subject to allowing for the offset NN', all setting out uses values of $T_{x'''}$ and $T_{y'''}$ given by (18) and it might be thought unnecessary to consider yet another rotation to determine the coordinates $T_{x''''}$ and $T_{y''''}$.

There is an important occasion when it is useful to know the x'''', y'''', z'''' coordinates of the shadow T. This is at the crucial stage when checks are made to ensure that the wall parameters are correct. There is no point in setting out the design if the shadow isn't going to appear where calculations suggest it should!

Notwithstanding any ideas the designer might have, the shadow of a nodus traces a constant-declination curve during the period of any particular day that the sun is shining on the dial. An obvious check is to note the actual positions of the shadow at intervals during the day and compare these with the positions determined by calculation.

Each position observed should be clearly marked on the wall, with a sequence number for reference purposes and the time of day recorded as UTC using a radio-controlled watch.

From an almanac (or otherwise) note the declination and equation of time for the date the observations were made. From this information, and allowing for the longitude difference from Greenwich, determine the local hour angle H of each observation. With this, and the declination δ and the wall parameters ϕ, A and ℓ, use (18) to determine the calculated position $(T_{x'''}, T_{y'''}, T_{z'''})$ of the shadow at each point.

Using a total station one can directly read off the observed position of the shadow at each point but it will be expressed using x'''', y'''', z'''' coordinates.

The coordinates of the shadow in the new coordinate system, $(T_{x''''}, T_{y''''}, T_{z''''})$, can readily be determined from the coordinates in the old system, $(T_{x'''}, T_{y'''}, T_{z'''})$, by multiplying by the appropriate rotation matrix:

$$\begin{pmatrix} T_{x''''} \\ T_{y''''} \\ T_{z''''} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\ell & -\sin\ell \\ 0 & \sin\ell & \cos\ell \end{pmatrix} \begin{pmatrix} T_{x'''} \\ T_{y'''} \\ T_{z'''} \end{pmatrix} = \begin{pmatrix} T_{x'''} \\ T_{y'''}\cos\ell - T_{z'''}\sin\ell \\ T_{y'''}\sin\ell + T_{z'''}\cos\ell \end{pmatrix}$$

Noting that $T_{z'''} = -n$:

$$\left.\begin{aligned} T_{x''''} &= T_{x'''} \\ T_{y''''} &= T_{y'''}\cos\ell + n\sin\ell \\ T_{z''''} &= T_{y'''}\sin\ell - n\cos\ell \end{aligned}\right\} \qquad (20)$$

Since the x'''- and x''''-axes coincide, $T_{x''''} = T_{x'''}$ but the expression for $T_{y''''}$ is of greater note. The factor $\cos\ell$ reflects the amount by which the 1 m separation of the horizontal lines of the square grid is reduced when considering their vertical separation. The term $n\sin\ell$ is the vertical separation of N and N', equal to $NN'\cos\ell$. The value of $T_{z''''}$ is not a constant (like $T_{z'''}$) but is not used.

Using (20), the *theoretical coordinates* $T_{x''''}$ and $T_{y''''}$ of each position can be calculated and then compared with the *observed coordinates*. Any systematic errors might be a consequence of poor estimates for A or ℓ.

Notes on the Final Rotation

The x'''', y'''', z'''' system of coordinates clearly coincides with the x'', y'', z'' system of coordinates and the rotation matrix used to determine (20) is the inverse of the rotation matrix used to determine (16). The coordinates of the sun S are indeed the same in the two systems of coordinates but the coordinates of the shadow T are not. In one case the shadow is falling on a vertical wall and in the other case it is falling on an inclined wall.

It is unlikely that a dial would be set out on a wall which inclined steeply forward so a large positive value of ℓ is not to be expected. A large negative value is very much a possibility and would apply to embankments, low sloping roofs and to horizontal sundials (where $\ell = -90°$).

In practice, no wall is truly vertical. Even when constructed to the highest standards a wall 10 m high may be out of true by 10 to 20 mm. If the sundial is large, even inclinations of this small order have to be taken into account.

In a particular case (the wall supporting the noon mark in Paternoster Square in London) the value of ℓ was estimated to be 0.0009 radians. Using a total station, the horizontal distance of the nodus from the wall $ON' = n'$ was estimated to be 4633.5 mm.

With an inclination as small as this, one may take $\cos\ell = 1$ so the nodus height n was also taken to be 4633.5 mm. The offset $NN' = n' \sin\ell$ is decidedly not negligible and using $\sin\ell = \ell$ this offset is $0.0009 \times 4633.5 \approx 4.2$ mm.

Point N, the foot of the perpendicular from the nodus to the wall, is over 4 mm above the horizon line which passes through N'. Point N' was used as the origin when setting out the design.

Using the measured values of ℓ and n', the position of the shadow can be expressed in x'''', y'''', z'''' coordinates by substituting in (20) as:

$$\left.\begin{aligned} T_{x''''} &= T_{x'''} \\ T_{y''''} &= T_{y'''} + 4.2 \\ T_{z''''} &= 0.0009\,T_{y'''} - 4633.5 \end{aligned}\right\} \qquad (21)$$

These expressions for $T_{x''''}$ and $T_{y''''}$ were used when the checks were made prior to setting out.

Using Iteration for Irregular Surfaces

Although the analysis above has considered the shadow of a point cast onto a plane, it can readily be adapted to determine the position of the shadow when the surface is irregular. Fig. 14 shows the wall of Fig. 13 replaced by an undulating surface.

To deal with such a surface, the x'', y'', z'' system of coordinates appropriate for a vertical wall are reused. The x''–y'' plane is vertical but the direction of the normal to this plane, the azimuth A of the z''-axis, is arbitrary except that it should be generally outwards from the dial surface. Natural symmetry in the surface may suggest a convenient value for A.

The goal, as usual, is to determine where the line from the sun S through the origin O intersects the surface. The point of intersection T is the position of the shadow. In Fig. 14, the sun S and the shadow T are shown as projections onto the y''–z'' plane.

Once the system of coordinates is established, the coordinates of the sun, $(S_{x''}, S_{y''}, S_{z''})$, are given by (12). The coordinates of the shadow, $(T_{x''}, T_{y''}, T_{z''})$, are given by (13) but n is now a function of x'' and y'' rather than a constant so (13) has to be adapted:

$$T_{x''} = -\frac{S_{x''}}{S_{z''}}\, n(x'', y'') \qquad T_{y''} = -\frac{S_{y''}}{S_{z''}}\, n(x'', y'') \qquad T_{z''} = -n(x'', y'') \tag{22}$$

Until now, the value of n has been the nodus height, the length of the perpendicular from the nodus to a plane surface defined by $z'' = -n$. With an irregular surface, the concept of the nodus height is adapted. Each point on the surface has its own nodus height which is the negative of the z''-coordinate of the point.

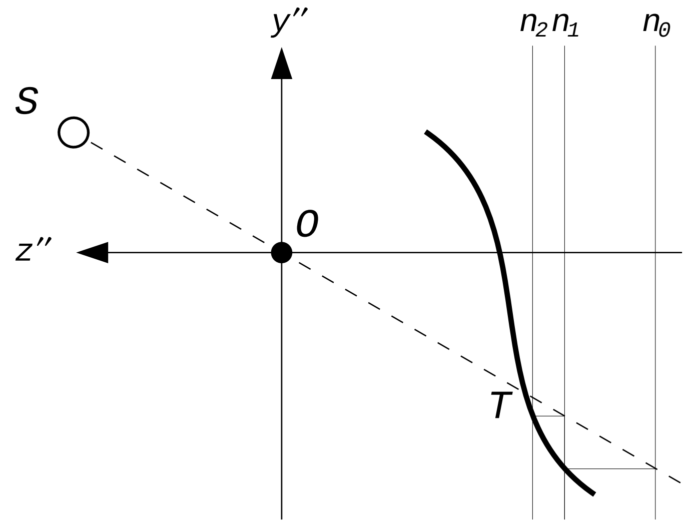

Fig. 14 — A Sundial on an Irregular surface

In outline, the procedure used to determine the position of the shadow it as follows:

1. Guess that the nodus height of the shadow T is n_0.
2. Use n_0 as the value of n in (22) to determine the coordinates $T_{x''}$ and $T_{y''}$ of the shadow if it were to fall on the $z'' = -n_0$ plane.
3. Determine $n_1 = n(T_{x''}, T_{y''})$, the nodus height of the point on the dial surface which has x''- and y''-coordinates $(T_{x''}, T_{y''})$.
4. Repeat from 2 using n_1 and continue this iterative process until successive values of n differ by less than, say, 0.1 mm.
5. Use the final value of n in (22) to determine close approximations to the coordinates $T_{x''}$ and $T_{y''}$ of the shadow on the dial surface.

Fig. 14 shows how successive iterations converge on point T but such convergence is not guaranteed. This procedure is not at all robust but it is satisfactory for near-planar surfaces which are close to the vertical (such as the wall supporting the noon mark in Paternoster Square). A binary-chop algorithm would be more robust but is not considered here.

Implementation of the Iteration in VBA

The iteration can be coded up in Visual Basic for Applications (VBA) and used from an Excel worksheet. Moreover, three VBA functions corresponding to the expressions for $S_{x''}$, $S_{y''}$ and $S_{z''}$ given in (12) can also be coded up for use from an Excel worksheet.

In outline, the three VBA functions which return the coordinates of S are:

```
Function xval(h, del, phi, A)
   xval = -Sin(h) * Cos(del) * Cos(A) + ...
End Function

Function yval(h, del, phi, A)
   xval = -Cos(h) * Cos(del) * Cos(phi) + ...
End Function

Function zval(h, del, phi, A)
   zval = Sin(h) * Cos(del) * Sin(A) + ...
End Function
```

In each case, the supplied arguments are H, δ, ϕ and A.

The iteration is also dressed up as a function which returns the approximate value of the nodus height:

```
Function nval(tanx, tany, n0)
   Nnew = n0
   Do
      Nold = Nnew
      Tx = Nold * tanx
      Ty = Nold * tany
      Nnew = nod(Tx, Ty)
   Loop Until Abs(Nnew - Nold) < 0.1
   nval = Nnew
End Function
```

The third argument is whatever guess is chosen for n_0. The first two arguments are:

$$\texttt{tanx} = -\frac{S_{x''}}{S_{z''}} \qquad \text{and} \qquad \texttt{tany} = -\frac{S_{y''}}{S_{z''}} \tag{23}$$

where $S_{x''}$, $S_{y''}$ and $S_{z''}$ are the values returned by `xval`, `yval` and `zval`. It is clear from Fig. 2 and Fig. 14 that the ratios in (23) are tangents which is why the identifiers `tanx` and `tany` are used. The coordinates $T_{x''}$, $T_{y''}$ and $T_{z''}$ are subsequently obtained by multiplying the result from `nval` by `tanx`, `tany` and -1 respectively.

The function `nod` is an implementation in VBA of the function $n(x'', y'')$ which first appeared in (22). This function is a description of the dial surface. For example, the dial side of the leaning wall in Fig. 13 could be described as:

$$n(x'', y'') = n' - y'' \tan(\ell) \tag{24}$$

Here y'' and z'' replace y'''' and z'''' with which they coincide.

It is instructive to consider a dial on a leaning wall as a surface with variable n to check that the iterative procedure works. The dial in Fig. 12 has a lean $\ell = 3°$ and a (constant) nodus height $n = 100\,\mathrm{mm}$. When this wall is treated as a surface with variable n the appropriate interpretation of (24) is:

$$n(x'', y'') = 100.137 - 0.0524078\, y''$$

Note that:

$$n' = 100/\cos(3°) = 100.137 \qquad \text{and} \qquad \tan(3°) = 0.0524078$$

Implemented in VBA the `nod` function could then be written as:

```
Function nod(x, y)
   nod = 100.137 - 0.0524078 * y
End Function
```

Recall that the coordinates $T_{x''}$, $T_{y''}$ and $T_{z''}$ which are obtained by multiplying the result of `nval` by `tanx`, `tany` and -1 respectively cannot immediately be compared with the coordinates $T_{x'''}$, $T_{y'''}$ and $T_{z'''}$ derived from (18). The coordinate systems are different!

To make the comparison, note that:

$$\left.\begin{aligned} T_{x'''} &= T_{x''} \\ T_{y'''} &= T_{y''}\cos(3°) + T_{z''}\sin(3°) \\ T_{z'''} &= -T_{y''}\sin(3°) - T_{z''}\cos(3°) \end{aligned}\right\} \tag{25}$$

All three of $T_{x'''}$, $T_{y'''}$ and $T_{z'''}$ determined from (25) should be within approximately 0.1 mm (the iteration limit) of the corresponding values of $T_{x'''}$, $T_{y'''}$ and $T_{z'''}$ derived from (18). In the present example, the nodus height in x''', y''', z''' coordinates is 100 mm so the value of $T_{z'''}$ should be within approximately 0.1 mm of -100.0.

The `nod` function is supplied with two arguments, `x` and `y`, but only `y` is used. The general surface will make use of both arguments.

Using iteration on a leaning wall is useful as a test but it is less efficient than using (18) because of the cost of the iteration steps.

Horizontal Dials — Arbitrary ϕ, $A = -180$, $\ell = -90$

The coordinates of the shadow given by (18) apply generally for valid values of H, δ, ϕ, A, ℓ and n subject to the sun being above the horizon and on the dial side of the wall.

Any wall for which $\ell = \pm 90°$ is clearly horizontal. If $\ell = +90°$ the dial is face downwards so the sun can never be above the horizon and on the dial side of the wall at the same time. To be useful, a horizontal dial has $\ell = -90°$.

In principle, the azimuth can be arbitrary but it is usually convenient to have the positive X-axis pointing east and the positive Y-axis pointing north. This is achieved if $A = 180°$. [If $A = 0°$ and $\ell = -90°$ the Y-axis will point south.]

Substituting $A = 180°$ and $\ell = -90°$ into (18) gives:

$$\begin{pmatrix} T_{x'''} \\ \\ T_{y'''} \\ \\ T_{z'''} \end{pmatrix} = \begin{pmatrix} \dfrac{\sin H \cos\delta}{\cos H \cos\delta \cos\phi - \sin\delta \sin\phi} \\ \dfrac{\cos H \cos\delta \sin\phi + \sin\delta \cos\phi}{\cos H \cos\delta \cos\phi - \sin\delta \sin\phi} \\ -1 \end{pmatrix} n \tag{26}$$

The sun is on the dial side of such a horizontal dial if it is above the horizon. Substituting $A = 180°$ and $\ell = -90°$ into (19) gives (10).

An alternative way of deriving (26) is to note from Fig. 15 that a vertical north-facing dial at latitude ϕ can be transferred with its orientation preserved to a point at latitude $90 - \phi$ but 180° round the earth.

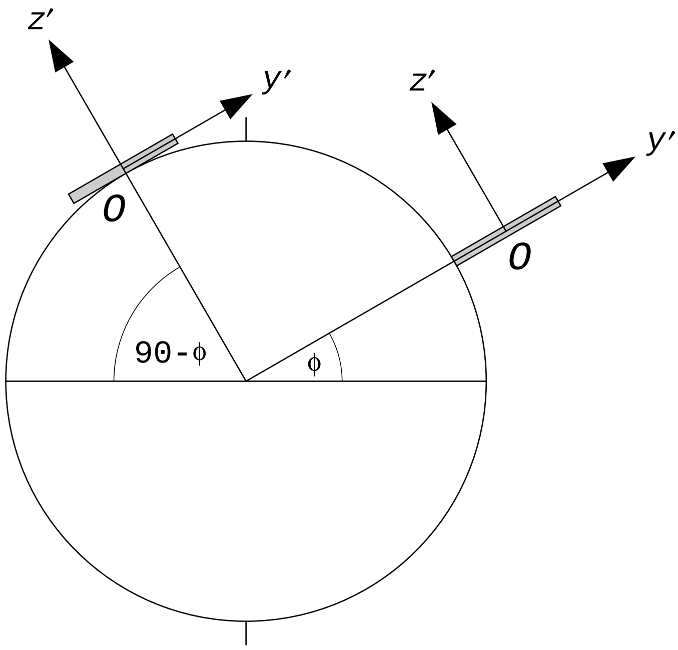

Fig. 15 — Projection of the Sun and the Shadow onto the z–x Plane

In its translated position, the x'–y' plane (the plane of the dial) is horizontal and the y'-axis points north as required. Note, though, that the hour angle in the new position is 180° different from before.

The coordinates of the shadow, $(T_{x'}, T_{y'}, T_{z'})$, for the vertical wall dial at latitude ϕ are given by (9). In the new position the expressions in (9) have to be adapted. Each instance of ϕ has to be replaced by $90 - \phi$ and each instance of H has to be replaced by $H + 180$.

In the trigonometric functions, $\sin\phi$ and $\cos\phi$ have to be changed to $\cos\phi$ and $\sin\phi$ respectively and $\sin H$ and $\cos H$ have to be changed to $-\sin H$ and $-\cos H$ respectively.

With these changes, the expressions in (9) become exactly as those in (26).

The Equinoctial Line — A Special Case

At an equinox $\delta = 0$ and the shadow of any point falling on a plane traces a straight line. This is readily demonstrated for the special case where the plane is vertical and $\ell = 0$.

Taking $\delta = \ell = 0$ and the foot of the perpendicular from the ball to the wall as the origin of an X, Y coordinate system (with X and Y parallel to x''' and y''' respectively), the upper two terms of (18) reduce to:

$$X = \frac{\sin H \cos A - \cos H \sin\phi \sin A}{\sin H \sin A + \cos H \sin\phi \cos A}\, n$$

$$Y = \frac{\cos H \cos\phi}{\sin H \sin A + \cos H \sin\phi \cos A}\, n$$

It can readily be demonstrated that:

$$Y = -\frac{\cos\phi \sin A}{\sin\phi}\Big(X - \frac{\cos A}{\sin A}\, n\Big) \tag{27}$$

in which ϕ, A and n are constants so Y is of the form $Y = mX + c$ and hence a plot of Y against X is a straight line.

Consider the right-hand side:

$$\text{RHS} = -\frac{\cos\phi \sin A}{\sin\phi}\Big(\frac{\sin H \cos A - \cos H \sin\phi \sin A}{\sin H \sin A + \cos H \sin\phi \cos A}\, n - \frac{\cos A}{\sin A}\, n\Big)$$

The expression in brackets reduces as follows:

$$= \frac{\sin H \cos A \sin A - \cos H \sin\phi \sin^2 A - \sin H \cos A \sin A - \cos H \sin\phi \cos^2 A}{(\sin H \sin A + \cos H \sin\phi \cos A)\, \sin A}\, n$$

$$= -\frac{\cos H \sin\phi}{\sin H \sin A + \cos H \sin\phi \cos A} \times \frac{n}{\sin A}$$

Now multiply by the omitted term to give:

$$\text{RHS} = \frac{\cos\phi \sin A}{\sin\phi} \times \frac{\cos H \sin\phi}{\sin H \sin A + \cos H \sin\phi \cos A} \times \frac{n}{\sin A}$$

$$= \frac{\cos H \cos\phi}{\sin H \sin A + \cos H \sin\phi \cos A}\, n = Y$$

This demonstrates the validity of (27).

Note from (27) that the equinoctial line has slope $-(\cos\phi/\sin\phi)\sin A$ and it happens that this is the same angle to the horizontal as the sub-style of a conventional gnomon makes to the midnight-noon line which is vertical on a sundial which is itself vertical.

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